OPTIMAL PORTFOLIO OF A SENSITIVE INVESTOR IN A FINANCIAL MARKET

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BY

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Abstract

A sensitive investor seeks to diversify assets and optimal portfolio which provide the maximum expected returns at a given level of risk. Optimal portfolio problems of an investor with logarithmic utility have been studied. However, there is scarce information on other utility functions, such as power utility function, which captures the concept of diversification of portfolios. This study was therefore designed to consider the general expected utility of a sensitive investor in a financial market.

Two models were derived from the Itô's integral with respect to power utility function. The extension of the Itô's integral by forward integral with its lofty properties was used to diversify the investors portfolio. A filtration was built and used as a set of information for the investor. A semimartingale was used to enlarge the investors information. A probability function was defined to capture the activity of an insider in the market and penalty function was established to punish such an insider. A priority Mathematical software was used to compute the investors varying rates of volatility.

The models derived were:

$$U'\left(S_{\beta_1\gamma_1+y\phi}(T)\right)S_{\beta_1\gamma_1+y\phi}(T)|M(y)| = S^y_{\beta_1\gamma_1+y\phi}(T)|M(y)|$$

and $n_t^{dis} = (1 - C_1C_2)(\rho_t^k + \pi_t)$, respectively, where $U'(x) = \frac{dU(x)}{dx}$ is satisfied if $\sup_{y\in(-\delta,\delta)}\{E[S_{\beta_1\gamma_1{}^y+y\phi}(T)|M(y)|^p] < \infty\}$ for some $p > 1$

$$0 < E[U'\left(S_{\beta_1\gamma_1+y\phi}(T)\right)S_{\beta_1\gamma_1+y\phi}(T)] < \infty$$

$$S_{\beta_1\gamma_1+y\phi}(T) = S_{\beta_1\gamma_1+y\phi}(T)N_{\beta_1\gamma_1}(y),$$

where

$$N_{\beta_1\gamma_1}(y) := s_0 \exp\int_0^T [\mu(s) - r(s) - \sigma^2(s)\beta_1(s)\gamma_1(s)]ds + \int_0^t \sigma(s)dW(s)$$

for all $\beta_1\gamma_1, \phi \in \mathcal{A}_{\mathbb{G}}$ such that $\mathcal{A}_{\mathbb{G}}$ is the set of admissible portfolios with diversification and ϕ bounded, then there was existence of $\delta > 0$ and $y \in (-\delta, \delta)$, where $W(t)$ is the Brownian motion (representing the fluctuation of the risky asset), on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t \geq 0, P)$ and the coefficients $r(t), \mu(t), \sigma(t)$ are $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$ adapted with $\mathcal{G}_t \supset \mathcal{F}_t$ for all $[0, T], T > 0$ a fixed final time.

The Itô's integral is adapted to the filtration $\mathcal{F} = \{\mathcal{G}_t\}_{0\leq t\leq T}$. The forward integral showed that when an investor buys a stochastic amount α units of this asset at some random time τ_1 and keeps all these units up to a random time $\tau_2 : \tau_1 < \tau_2 < T$, and eventually sells them at a subsequent time, the profits realised would be $\alpha W(\tau_2) - \alpha W(\tau_1)$ expressed as forward integration of the portfolio $\phi(t) = \alpha I(\tau_1, \tau_2](t),\ t \in [0, T]$ with respect to the Brownian motion $W(t)$ i.e.

$$\int_0^T \phi(t)d^-W(t) = \lim_{\Delta_j \to 0} \sum_j \phi(t_j) \times \Delta W(t_j) = \int_{\tau_1}^{\tau_2} dW(t) = \alpha W(\tau_2) - \alpha W(\tau_1)$$

The filtration $\mathcal{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$ outlined the information flow of the investor. The semimartingale integral $\int_0^T \phi(t)dW(t) = \int_0^T \phi(t)d^-W(t)$ gives a decomposition $W(t) = \hat{W}(t)+A(t),\ 0 \leq t \leq T$, where $\int_0^T \phi(t)dW(t) = \int_0^T \phi(t)d\hat{W}(t)+\int_0^T \phi(t)dA(t)$; for $\mathcal{G}_t = \mathcal{F}_t \vee \alpha W(T_0);\ 0 \leq t \leq T$ i.e. $\mathcal{G}_t$ is the result created by $\mathcal{F}_t$ and the final value $W(T_0)$, where $\hat{W}(t)$ is a $\mathcal{G}_t$-Brownian motion and $A(t)$ is a continuous $\mathcal{G}_t$-adapted finite variation process. The probability of detecting and punishing an insider was $\lambda_1 = 1$ and λ_2 showed the penalty on an insider observation. The varying rates of volatility $\sigma = 1, 0.5,\ s_0 = 100, \mu = 1$, revealed that the expected return is more when volatility $\sigma = 1$, thereby yielding optimal portfolio.

The optimal portfolio of a sensitive investor was established using power utility function and showed higher investors return as the investor diversified his investment.

Keywords: Power utility function,Diversification, Itô-integral, Semimartingale.

Word count: 464

Dedication

To GOD from whom all blessings flow and my Late Mother Mrs Comfort Achudume Osakwe who painstakingly invested in my education, I dedicate this thesis in remembrance of your love and confidence in me. Mummy the memories of your care still linger in me.

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dream come through will remain indelible in my heart May God remember you all in Jesus name Amen.

Certification

I certify that this work was carried out by **Mr Celestine Achudume** with matriculation number 152717 in the Department of Mathematics, Faculty of Science, University of Ibadan under my supervision

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The following notations were used for this work.

$\beta_1\gamma_1$ represents assets in a portfolio

σ, ω represents stock price volatility

μ, ϑ represents the drift term

ρ, λ represents the risk-free interest rate

$P \sim Q$ represents equal measure of P with Q

$H_m[F]$ represents generalised expectation with respect to measure m

$H_Q[F]$ represents generalised expectation with respect to measure Q

$H[F|\mathcal{F}_t]$ represents generalised conditional expectation

$a.a, a.e., a.s.$ represents almost all, almost everywhere, almost surely

$\mathcal{A}_{\mathbb{N}}$ represents sets of admissible process

$W = W(t) = W(t, w), D(t), B(t)$ represents the process of Brownian motion

$\mathcal{F}_t, \mathcal{N}_t, \mathbb{N}$, represents filtrations

$(\Omega, \mathcal{F}, \mathcal{P})$ represents probability space

$dt, d\tau, ds$ represents Lebesgure measure

Chapter 1
INTRODUCTION

1.1 Background

The present economic situation in Nigeria means that reliance on a single source of income can no longer satisfy the needs of an average middle class family. Thus, the need to explore multiple streams of income is on the front burner of many Nigerian homes. Possible means of achieving multiple streams of income, but better means is to invest in an asset and even better means is to invest in multiple assets. In economics, we know that man's needs are insatiable, thus man always seeks for means to increase his expected financial returns.

A person who seeks to put finances into the acquisition of an asset in other to get higher returns is called an investor. Generally, investors are classified into three categories, namely: risk neutral, risk averse and risk seeking. Risk-averse is someone who avoid taking risk thus so conservative, risk-taker on the contrary is ready to take more risk hoping to gain more return and risk neutral also called risk-indifferent who is neither risk-averse nor risk-taker. The investors attitude to risk determines his investment preference, for example, a risk averse investor would prefer to put his money into a bank account, invest in bonds or put his money into a fixed deposit. Also, the expected returns of an investor is determined by his attitude to risk. The expected returns of a risk seeking investor is always higher than that of a risk averse investor. However, the risk averse investor has a chance of getting his expected returns through diversification.

A sensitive investor is an investor who has the means and seeks to diversify investment. He or she desires the most favorable portfolio which provides maximum expected returns at a specified level of risk. A portfolio is a group of assets. Thus, the investor considered in this study is one who wants to be engage in both small and multiple investment. Many financial institutions such as banks, insurance, commercial institutions, real estates and public sectors who opt for higher returns at acceptable risk levels are in these category of investors and are referred to as sensitive investors. This is because some banks are into hostel and residential developing contracts with universities and big organisations to be managed before handing over.

Risks occur as a result of data that are uncertain. Risk is also seen as the likelihood of actual future returns varying from the expected returns. However, from the context of security and investment analysis, risk is also the likelihood that real cash flows will be different from the estimated cash flows Moyer *et al.* (2006). Risk and expected returns are intrinsic part of investments and are treated concurrently. A sensitive investor seeks and resorts to diversification so as to spread investment and reduce risk. A diversified portfolio or group of assets has a smoother risk behavior, that is, it is a much more robust investment option Derman (1994).

Diversification aims to reduce the unsystematic risk in an investment portfolio which occur as a result of mismanagement, poor forecasting accuracy or wrongful planning process and decision making. Diversification helps to reduce the volatility of portfolio performance. This is because holding diverse assets implies that the price of diverse assets does not change in the same direction, at the same time or at the same rate. Thus, diversified portfolio is more robust with less variation in expected return.

Good portfolio management entails active investment and paying close attention to market trends against spontaneous shift in the economy and changes to the political landscape as well as factors that may affect some organisations. This enables monitoring transactions that could affect the price of assets in other to take decisions that might boost higher returns and hedge against loss. For example, the risk of exchange rate variance significantly affect the utilities and portfolio choices of both domestic and foreign investors. As a result, the variance and correlations in returns are unpredictable Mandelbrot (1963), therefore, there is the need to hedge risk against any unforeseen circumstances.

In the business environment, good investment performance is primarily determined by the quality of the investment decisions, which are invariably made in an environment of uncertainty concerning technology, market-places, competitors, legal issues, and so on. Sub-optimal decisions can hamper a business, and sometimes cause serious damage. In many cases, sub-optimal decisions are made because the right information is not available or because there is not enough data. However, in some cases the fault lies in the method used to arrive at the decision Manganelli *et al.* (2014). Therefore, what determines an investors success are well-considered decisions and good risk management as well as exhaustive research on every possible scenario and options available for increasing investors profit.

Many studies have considered optimal portfolio with logarithmic utility. However, logarithmic utility which considers present investment opportunities does not fully capture the concept of diversification, therefore, we can say it is short-sighted or myopic. However, power utility considers future investment opportunities and is thus a more robust option than logarithmic utility.

1.2 Motivation for the study

Generally, an investor desires maximum expected returns on his investment. Thus, investors are always and will always be on the lookout for means that guarantee greater return on investments while hedge against risks. Many studies have considered investors portfolio using logarithmic utility. However, logarithmic utility does not keep an eye on future investment opportunities and is thus myopic. Thus, it does not adequately take care of diversification of assets which is the crux of the matter for a sensitive investor.

The sensitive investor makes use of the information he or she has to hedge against future risks. This motivates us to look at a different utility which adequately takes care of the concept of diversification of assets for the sensitive investor.

1.3 Statement of problem

The Optimal portfolio problems of a sensitive investor is the subject matter of this research work. Optimal portfolio problems of an investor with logarithmic utility have been studied. Logarithmic utility does not capture the concept of diversification. Hence, it is referred to as myopic. However, there is scarce information on other utility functions, such as power utility function, which captures the concept of diversification of portfolios. This study was therefore designed to consider the general expected power utility of a sensitive investor in a financial market.

1.4 Research aim and objectives

This research studied a sensitive investors portfolio which yields maximum expected returns by considering a general expected utility of a sensitive investor. The objectives are:

1. To provide measures of reducing risk in investment via diversification and specification of small scale investors
2. To provide multiple assets diversification with restrictions for investors on the quantity of assets to be held due to transaction cost and risks.
3. To provide characteristics of optimal portfolio of a sensitive investor with insurance cover
4. To provide effective risk management of portfolio investment through assets diversification for large scale investors under insurance cover
5. To provide measures of curbing information asymmetry in the market.

1.5 Research methodology

A model was derived from the It$\hat{o}$'s integral with respect to power utility function. The extension of the It$\hat{o}$'s integral by forward integral with its lofty properties was used to diversify the investors portfolio. A filtration was built and used as a set of information for the investor. A semimartingale was used to enlarge the investors information. A probability function was defined to capture the activity of an insider in the market and penalty function was established to punish such an insider. A priority Mathematical software was used to compute the investors varying rates of volatility. The filtration, $\mathbb{N} = \{\mathcal{N}_\tau\}_{0 \geq T}$ is slightly bigger than $\mathbb{F} = \{\mathcal{F}_\tau\}_{0 \leq \tau \leq T}$ of $D(\tau)$. A sensitive investors business logistics is $\varpi(x, w) = I_{\{\tau_1 < x \leq \tau_2\}}$ and considered as a buy-and-hold strategy that is, we assume $D(\tau)$ is the risky assets price at time τ, if an investor purchases the fluctuating amount α units of such asset at time τ_1 of unpredictable process and eventually decide to keep them to a certain time $\tau : \tau_1 < \tau_2 < T$, and sell them at a subsequent time, the profit realised would be $\alpha D(\tau_2) - \alpha D(\tau_1)$

$$\varpi(\tau) = \alpha I(\tau_1, \tau_2](\tau), \ \tau \in [0, T]$$

with respect to $D(\tau)$

$$\int_0^T \varpi(\tau)d^- D(\tau) = \alpha D(\tau_2) - \alpha D(\tau_1),$$

where τ_1, τ_2 are bounded random times.

In as much as $\varpi(t)$ is forward integrable with respect to $D(t)$, then $\int_0^T \varpi(t)dD(t)$ is a semimartingale integral and

$$\int_0^T \varpi(t)dD(t) = \int_0^T \varpi(t)d^- D(t), \tag{1.1}$$

where ϖ is adapted to $\mathcal{N}_t \supset \mathcal{F}_t$, and such that $D(t)$ is a semimartingale with respect to $\mathcal{N}_t$ filtration and with a decomposition

$$D(t) = \hat{D}(t) + A(t), \ \ 0 \leq t \leq T, \tag{1.2}$$

where $\hat{D}(t)$ is a $\mathcal{N}_t$- Brownian motion and $A(t)$ is a continuous $\mathcal{N}_t$-adapted finite variation process. If $A(t)$ has the form

$$A(t) = \int_0^t \alpha(s)ds, \tag{1.3}$$

then the process $\alpha(.)$ is the information drift. In general, if a relation of the form (1.2) holds, then we define

$$\int_0^T \varpi(t)dD(t) = \int_0^T \varpi(t)d\hat{D}(t) + \int_0^T \varpi(t)dA(t). \tag{1.4}$$

Let $T \leq T_0$ and

$$\mathcal{N}_t = \mathcal{F}_t \vee \omega D(T_0); \ 0 \leq t \leq T \tag{1.5}$$

that is, $\mathcal{N}_t$ is the ω-algebra produced by $\mathcal{F}_t$ and the final value $D(T_0)$, where T_0 is the final time. Then

$$\hat{D}(t) := D(t) - \int_0^t \frac{D(T_0) - D(s)}{T_0 - s} ds; \ \ 0 \leq t \leq T \tag{1.6}$$

is an $\mathcal{N}_t$- Brownian motion, as such (1.2) is established with

$$A(t) := \int_0^t \frac{D(T_0) - D(s)}{T_0 - s} ds; \ \ 0 \leq t \leq T. \tag{1.7}$$

By (1.4), we get

$$\int_0^T \varpi(t)d\hat{D}(t) + \int_0^T \varpi(t)dA(t) = \lim_{\Delta\tau_j \to 0} \sum_j \varpi(t_j)\Delta D(t_j)$$

$$= \int_{\tau_1}^{\tau_2} dD(s) = D(\tau_2) - D(\tau_1)$$

$$= \int_0^T \varpi(t) d^- D(t)$$

and

$$\int_0^\infty \varpi(s) d^- D(s) = \lim_{\delta \to 0} \int_0^\infty \varpi(s) \frac{D(s+\delta) - D(s)}{\delta} ds$$

Given that ϖ is *cáglád* and continuous on the left with right limits, then

$$\int_0^s \varpi(s) d^- D(s) = \lim_{\Delta_{sj} \to 0} \sum_j \varpi(s_j) \times \Delta D(s_j)$$

We have this as

$$\varpi(s) = \sum_{j=1} \varpi(s_j) I(s_j, s_{j+1}](s)$$

then

$$\int_0^\infty \varpi(s) d^- D(s) = \lim_{\delta \to 0} \int_0^\infty \varpi(s) \frac{D(s+\delta) - D(s)}{\delta} ds$$

$$= \sum_{j=1}^n \varpi(s_j) \lim_{\delta \to 0} \int_{s_j}^{s_j+1} \frac{D(s+\delta) - D(s)}{\delta} ds$$

$$= \sum_{j=1}^n \varpi(s_j) \lim_{\delta \to 0} \frac{1}{\delta} \int_{s_j}^{s_j+1} \left(\int_s^{s+\delta} \right) ds dD(s)$$

$$= \sum_{j=1}^n \varpi(s_j) \lim_{\delta \to 0} \frac{1}{\delta} \int_{s_j}^{s_j+1} \delta dD(s)$$

$$= \sum_{j=1}^n \varpi(s_j)(D(s_{j+1}) - D(s_j)).$$

Assuming optimal sensitive investors portfolio $\nu^*(t)$ exists, then $\nu^*(t) = \beta_1 \gamma_2$ equally exists implying that if the investor diversify his investment by leasing to yield huge fund, where $\varpi = \omega(t)\nu^*(t)$. Our analysis have shown that the higher the drift and volatility, the more the expected return which represents a positive investment. It is also observed that the expected return on the utility function of an investor investing $\alpha = 2p - 1$ of his money when the probability of realising a positive expected return is more than 0.5.

1.6 Structure of the study

The structure of this thesis is highlighted as follows. In Chapter One, we present the introduction. In Chapter Two, we present the literature review. Chapter Three consists of relevant mathematical preliminaries for this study. The lofty properties of forward integral, which is an anticipating integral couple with its techniques and the implications of using this model were presented. In Chapter Four, the variational method of a sensitive investor's optimal portfolio were presented. Finally, in Chapter five, we present our conclusions and further research focus.

Chapter 2
REVIEW OF LITERATURE

The choice of portfolio is a classical issue in mathematical finance, where the principal intention of any investor is to seek for the best quantity of money to invest in the risky assets for a benefit. The establishment of optimal portfolio constitute some difficulties for market partakers since they operate in an unpredictable environment.

Harry Markowitz (1952), initiated the mean-variance (MV) model, developed the choice of portfolio problem as an optimisation problem, that is made up of minimising the variance (an investors risk measure) of the final value for a wished amount of expected return. He considered highest expected return with a minimum variance in return, which is not always the case.

Michaud (1989), observed that the mean-variance optimised portfolios are difficult to comprehend and made it difficult for investors to practice easily. However, in the actual sense, the Markowitz formulation fail to consider the investors consumption.

Another interesting area of study in the sense of portfolio optimisation is concerned with the utility theory as well as expected utility maximisation, where the likes, preferences and desires of investors are expressed by the utility function. In this sense, investors goal is to maximise the expected value of the utility function..

The general continuous-time issue of optimal consumption and choice guideline of portfolio was considered first by Merton (1971), and established the concept for dynamic portfolio selection under unpredictable scenario. Mertons concept of dynamic programming inspired a partial differential equations (PDE) of nonlinear which is complex and a general issue considering the process controlling the volatility; this appearance makes it challenging to determine the best approach for the portfolio and ideal consumption. However, Merton (1967) clearly proffer a PDE based solution under a consistent risky asset volatility.

Up to date, researchers have developed and sustained huge interest on portfolio optimisation problems with stochastic volatility. Fleming *et al.* (2003) resolved the long standing issue of both the consumption and the choice of portfolio simultaneously, where the driven stochastic volatility is related to the diffusion process. In contrast Goll and Kallsen (2003), obtained clear results for portfolios exhibiting log-optimal in a complete market with semimartingale specification of the price process.

Kramkov *et al.* (1999) considered arbitrage-free model where relevant results for the problem of optimal investment was established. Chacko and Viceira (2005), achieved more promising result for an incomplete market driven by the CIR model (1985). Bae *et al.* (2015), established a time-varying volatility model of a stock market through the process of regime switching method and a constant interaction influence.

However, an issue synonymous with that of Merton was resolved by Brennan (2000),(2001),(2002) and Wachter, (2003). Liu, (2007) introduced a clear solution to the problem choice of portfolio dynamics, where the risky assets return are controlled by a "quadratic process", which takes the form of Markovian diffusion process as well as a consistent relative risk aversion (CRRA) of investors utility function. Coulon (2009), numerically solved the Merton model with a robust finite difference method. He admitted that the condition of convergence were looked upon, and the method proposed needed to be recast for the time-discretisation to converge. The revisiting of the portfolio choice problem and optimal consumption of an investor with access to a risk-free asset and with a consistent expected return and stochastic volatility. His intention was double, first he determined a detailed portfolio dynamics solution as well as the issue of selection, when the returns of the risky assets is volatile and controlled by priority process of Ornstein-Uhlenbeck, for an investor under (CRRA). Secondly, he computed several numerical test with the aid of the obtained solution to be able to resolve the optimal amount that is sensitive as well as the consumption and several variables in the model. The risky asset expected return, investors risk-averse, the force of mean-reverting, the long-term mean together with diffusion coefficient of the stochastic influence of Brownian motion was also considered.

Russo *et al.* (1993),(1995),(2000) considered the forward backward stochastic integral and introduced a larger class of gaussian finite variation process as well as approximating the integrator. Kreps (1981) introduced the exclusion of arbitrage behavior as a requisite and sufficient criterion for economic balance. Alexander *et al.* (2004), (2006) considered the European pricing and hedging of options spread of assets correlation as the return of each assets marginal distribution exhibit normal distribution mixture. Becker (1980) established a constant variance elasticity as the main behavior of stock price than lognormal model. Bingham (2004) considered the risk-neutral model as the most efficient numiraire for pricing option. Brigo(2002), (2003), (2004). considered the densities of log-normal allowing several means as well as hyperbolic-sine type processes.

Elliot, (1991) introduced the incomplete market as a result of incomplete information which generated unequal martingale measure, and thought of minimising the martingale measure $\hat{P}$ to preserving the structure of the market. The use of the non restrictiveness in making speculations over μ and σ for price, and the choice of σ and μ be made strong to abate arbitrage, and introduced $\mu = \frac{1}{2}$ where $\sigma = 1$, which implies knowing the maximal stock price in advance was introduced by Imkeller *et al.* (2001).

Two investors with logarithmic utility functions comprised of one who make his portfolio decisions based on the available information while, the other possesses additional information was considered by Amendinger *et al.* (2003). It was observed that unequal information between investors in the market does not forster unity and discourages both local and foreign investors as a result, Corcuera *et al.* (2004), established equal information for every trader in the market due to deformation of insiders knowledge. He obtained, a semimartingale decomposition of the process and observed that the slow vanishing of the process confirms no arbitrage and shows the finiteness of the insiders additional utility. Leon *et al.* (2003), established a vital observation of an investor possessing extra information of the future development of the market known as an insider of the market. Ackerman *et al.* (2015), introduced insider trading laws and regulations of investors in the market. Karatzas *et al.* (1996), established a penalty function $f(x) := -Cx^{\gamma}, 0 < \gamma < 1$ and $C > 0$ where C is relative to the crime, γ , represents the law and enforcement agencies.

Heston *et al.* (1993), studied a stochastic volatility with simulation showing assets price relevance for explaining returns. Applebaum *et al.* (2009), introduced a purely discontinuous model process since the returns of assets are usually defined as increments of *log* stock prices, that is $\log \hat{L}_t - \log L_{t-1}$ where

$$\hat{L}_t = L_0 \exp(X_t)$$

with $X = (X_t)_{t\geq 0}$ as the corresponding return process. Kaur *et al.* (1994), established the dominance of an investment that guaranteed a higher return with higher risk over an investment with high utility rate and lower risk.

Anderson *et al.* (2013), obtained Optimal returns in real estate investment as much effort of the brokers. The brokers through the media highlight those features that satisfies the customers. Such as the architecture and high technological design, urban planning, quality of the luxury. During marketing process however, strategies that require long time payment with higher price than the normal market price would be initiated. Amunuay *et al.* (2016), considered some parameter λ in the Black-Scholes equation which depends on the interest rate ρ and volatility σ while λ changes as the value of ρ and σ equally varies. Meilling Deng (2015), studied sufficient conditions for stochastic permanence, extinction, global attractiveness and stability in a stochastic competitive model with Markov switching.

Qiang Li (2015) considered a risk-averse firm which procures some kind of commodity from the spot market as a major input for production. Nualart *et al.* (2004) introduced a restrictions on the integrands in Itô stochastic integral for the fact that the measurability condition which prescribes that the integrand is independent on the future increments of the integrator.

Ali *et al.* (2016) introduced the traditional portfolio selection problem geared by establishing four constraints such as collinearity meant to decrease the portfolio risk, active stock to regulate risk, as well as to increase the un-expected return which is the course of inefficient market and finally to control the net risk of the portfolio.

Cox-Ross *et.al* (1985), considered logarithmic utility, where the optimal size of money on stock absolutely rely on the model parameter. The optimal portfolio with logarithmic Utility disregards the differences between the present value and influence of the future of an economy, as a result, it is myopic. Because,the size of wealth owned in stock appears to be the same throughout the periods of time, even at the random change of variables of the market. Furthermore, power utility functions would consider future investment opportunities. For example, if the current interest rate of the risky assets is presumed higher than the future, a sensitive investor might be receptive to the risky assets investment to take advantage of its unpredictable drop in the future. In this respect, we look at the general expected utility function which maximise a sensitive investors portfolio. Our model is built from Biagini *et al.* (2008) and Nualart *et al.* (2003) whose aim was to take undue advantage of the more acquired information to maximise the expected logarithmic utility from the final amount, we are using forward integral which is an anticipating integral to maximise the general expected utility from the final wealth of a sensitive investor whose intention is to diversify investment with the information he or she has.

Chapter 3
METHODOLOGY

3.1 Introduction

Relevant mathematical tools used in this study were discussed in this chapter. Firstly, we consider the background of stochastic calculus consisting of definitions and some concepts of a financial market. Thereafter, we discuss utility functions with emphasis on power utility functions. We conclude this chapter by discussing the forward integral.

3.2 Some background of stochastic calculus

The procedures for valuation of derivatives is centered on a vital specification of a stochastic process for the underlying assets. Stochastic calculus is essential in the analysis of such processes. Here we discuss some important ideas on stochastic analysis and market theory that show the modern approach to the mathematical modelling and pricing of contingent claims. Wilmott (1998) and Tan (2006).

Definition 3.2.1 Measurable space (Ω, β)

A pair (Ω, β) comprising a sample space Ω and a σ-algebra of subsets of the sample space Ω, is called a measurable space and a member of β is a measurable set.

Definition 3.2.2 A measure μ

Let a measurable space be (Ω, β). Then a map $\mu : \beta \to \mathbb{R}_+^* = [0, \infty) \cup \{\infty\}$ is called a measure provided that

(i) $\mu(\phi) = 0$.

(ii) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for any pairwise disjoint sequence of members of β.

Definition 3.2.3 A measure space

Let (Ω, β) be a measurable space and μ a measure on β. Then the triplet (Ω, β, μ) is a measure space.

Remark 3.2.3.1

1. A measure space (Ω, β, μ) such that $\mu(\Omega) < \infty$ is finite and μ is a finite measure

2. We assume all measure spaces are finite.

Definition 3.2.4 A complete measure space

Given (Ω, β, μ) a probability space where $\tilde{n} \in \beta$ and $\hat{m} \in \beta$, $\mu(\hat{m}) = 0$ and $\tilde{n} \in \hat{m}$, that is, all subsets of a set when measured gives zero and are measurable, then μ is called complete and (Ω, β, μ) is called a complete measure space.

Definition 3.2.5 Space of random variable

A random variable assigns to each elementary event ω a real value (or vector of values). In our context the random variables mostly stand for payments or values of financial products depending on the state of the world.

A random variable X is a $\mathcal{F}$-measurable function defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ mapping its sample space Ω into the real line $\mathbb{R}$:

$$X : \Omega \to \mathbb{R}.$$

Since X is $\mathcal{F}$-measurable we have

$$X^{-1} : \beta \to \mathcal{F}.$$

A random variable X is $\mathcal{F}$-measurable if the value of X is completely determined by the information in $\mathcal{F}$. Formally speaking, a map $X : \Omega \to \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called $\mathcal{F}$-measurable if

$$X^{-1}(U) = \{\omega \in \Omega : X(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \in \mathbb{R}$. A probability space is a triplet (Ω, β, μ) comprising of a set Ω, a σ-algebra β of subsets of Ω and a measure μ on the measurable space such that $\mu(\Omega) = 1$.

Remark 3.2.5.1

Let (Ω, β, μ) be a probability space. Then Ω consists of the set of sure outcomes. The members of Ω are called sample points or an elementary event, while a member of β is the event. If $A \in \beta$, then $\mu(A)$ means event A will occur.

Definition 3.2.6 Filtration

From our remark above (Ω, β, μ) and $\mathbb{F}(\beta) = \{\beta_t : t \in [0, \infty)\}$ a collection of sub σ-algebra of β satisfying:

(i) β_0 contains all the μ-null members of β.

(ii) $\beta_s \subseteq \beta_t$ for $t \geq s \geq 0$;

(iii) $\mathbb{F}(\beta)$ is right-continuous for $\beta_{t+} = \beta_t, \;\; t \geq 0$ where

$$B_{t+} = \bigcap_{s>t} \beta_s$$

then $\mathbb{F}(\beta) = \{\beta_t : t \in [0, \infty)\}$ is called a filtration of β and $(\Omega, \beta, \mu, \mathbb{F}(\beta))$ is the stochastic basis or a filtered probability space.

A filtration or information flow on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is an increasing family of σ-algebras $\mathcal{F}_t \; t \in [0, \infty)$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. $\mathcal{F}_t$ then is the current knowledge or information at t, and does grow as time progresses. Suppose the set of possible events is $\mathcal{F}$, in that sense $\mathcal{F}_t \subseteq \mathcal{F}$. We denote the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ by the symbol $\mathbb{F}$.

A probability space which is equipped with a filtration $\mathbb{F}$ is called a filtered probability space and denoted by $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$. One can intuitively say that The probability of the occurrence of a random event will changes as more information is revealed with time.

Filtration explicitly explain the flow of information which explains those things that are known based on the present information from those things regarded as random at a specific time t.

A rational investor would automatically ascertain if event $A \in \mathcal{F}$ has occurred or not based on the available information $\mathcal{F}_t$. Assuming the value of X is explicitly unveiled at time t or X is totally known by the information $\mathcal{F}$, then X is $\mathcal{F}$-measurable random variable.

Definition 3.2.7 Adapted process

A stochastic process $(X_t)_{t\in[0,\infty)}$ is said to be $\mathcal{F}_t$-adapted with respect to the information structure $(\mathcal{F}_t)_{t\in[0,\infty)}$ if for each $t \in [0, \infty)$, the value of X_t is revealed at time t, the random variable X_t is $\mathcal{F}_t$-measurable. An adapted process is also called a non-anticipated process.

3.2.1 Stochastic processes

A stochastic process is a set of random variables indexed by t. A family $\{X_t : t \geq 0\}$ of $\mathbb{R}$-valued random variables parameterised by time $t \in [0, \infty)$ stated on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is a stochastic process Shreve (2004). However, t can either be discrete or continuous. In any occurrence of $\omega \in \Omega$ randomness, there is a trajectory part $t \to X_t(\omega)$ which assume function of time as well as the sample path of the process. We assume this sample paths are right continuous. The index t is the time, we consider the fact that events become less unpredictable as several information on that events becomes available as time progresses Schwert (1989). It becomes vital to describe how several information about the event is unveiled.

However, achieving this is by introducing the idea of a filtration in which other important concept such as past information, predictability and adaptiveness of the process Francesco *et al.* (2014) and Fouque *et al.* (2000). A filtration is the information or knowledge of a stochastic process at a certain time.

3.2.2 Markov processes

A Markov process is a special type of stochastic process where only the present value of a variable is necessary for predicting the future. The past history of the variable and the way the present emerged from the past is irrelevant Howard (1971), Kemeny (1974) and Schonbucher (1999). A Markov process is a stochastic process for which the future does not rely on the past, but only the present. It is the building block of many stochastic processes, such as some path-dependent stochastic processes.

Definition 3.2.2.1

A stochastic process $(X_t)\, t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called a Markov process if:

$$\mathbb{P}\left(X_t \leq x | x_u\right) = \mathbb{P}\left(X_t \leq x | X_s\right)$$

for $0 \leq u \leq s \leq t$. A sequence with the Markov property is a Markov chain in the sense of discrete-time. Diffusion is often used in finance.

A diffusion process is equally a Markov process whose paths are consistent with time Dumas (1998). It generalises the Brownian motion, which accommodate wider class of events to be studied. Another interesting example of a Markov process is the Poisson process, in that predicting the future should not affect the price of assets a week ago, neither a month ago or a year ago. The necessary information needed is the price now. Because predictions are unpredictable it is best expressed in the form of probability distributions Cont (2004). The Markov property entails that the probability distributions of the price at the future time does not dependent on the specific path the price followed in the past.

3.2.3 Brownian motion

The Brownian motion is another example of stochastic process often used for modelling of stock price fluctuations. The paths of a Brownian motion are continuous but very erratic. There are some interesting properties of Brownian motion. A Brownian motion is also a Markov process. This implies that the future value of the probability distribution process depends solely on its present value and does not dependent on the past values.

Therefore, only the present process values information is required in making prediction about the future. Also, even though a Brownian motion is continuous, it is no where differentiable. Furthermore, a Brownian motion does not have parts of bounded variation, i.e. a Brownian motion can assume any real value irrespective of how big or non-positive the values are. $\psi = \{\psi \geq 0\}$ with its definition on $(\Omega, \mathcal{F}, \theta)$ is Brownian satisfying the following:

(i) The trajectory of ψ are θ-a.s are incessant and unending.

(ii) $\theta(\psi_0 = 0) = 1$.

(iii) $0 \leq s \leq t$, $\psi - \psi_s$ has similar distribution with ψ_{t-s}.

(iv) For $0 \leq s \leq t$, $\psi - \psi$ does not dependent on $\{\psi \leq s\}$.

(v) For each $\tilde{t} > 0$, ψ has equal distribution as normal random variable with variance $\check{t}$.

3.2.4 Martingale

Martingales are necessary and useful in the stochastic process study. A stochastic process in the theory of probability is a martingale if its sample path does not have any trend. That is, a stochastic process whose future movements are uncertain.

It implies a fair game modelling, martingale strategy requires a gambler to double his bet after losing. In line with this martingale strategy, a gambler can regain his earlier losses as well as his initial amount of wealth couple with an initial bet Musiela (2005). Therefore, the strategy of martingale implies that after several gambling, gains or lose nothing and have a constant wealth on the average.

Furthermore, martingale is absolutely a random process such that, observing the process history, the expected value of the process at a later time is its present value. i.e. $E(M_t) = E(M_0)\ \forall t \geq 0$ It is equally seen as a constant on average shown from our third condition below: the easiest forecast of unknown future value M_t relying on the current time s information, $\mathcal{F}_s$, is the value M_s known at time s. Therefore, the expected value of a martingale M at some time T (based on the initial information at time 0) equals its initial value M_0

$$E[M_T|\mathcal{F}_0] = M_0$$

Definition 3.2.4.1

A process $(M_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is a martingale with respect to $\{\mathcal{F}_t\}t \geq 0$ filtration, such that $\mathcal{F}_t \subset \mathcal{F}$ satisfying the following conditions:

1. M_t is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ i.e. M_t is $\mathcal{F}_t$-measurable for all t
2. $E[|M_t|] < \infty$, for all $t \geq 0$.
3. $E[M_t|\mathcal{F}_s] = M_s$ for all $0 \leq s \leq t$ P- a.s. replacing the equality by $\leq$(respectively)$\geq$, then M_t becomes a supermartingale (respectively submartingale).

i.e. sighting the future value based on the current information is not more than the current value.

As the time goes, a submartingale gains or grows on the average. A supermartingale is a stochastic process with the (downward) trend of negativity. A supermartingale does not gain on average as time goes. A supermartingale and a submartingale is also a martingale.

However, the random process that models the principle of randomness in the real sense is a martingale. unarguably, a martingale means there should be no future outcome prediction of events.

In this sense, assuming a stochastic process is a martingale, then its best future values prediction is its present value. To ensuring the conditional expectation existence, the finite mean condition is necessary. Remembering the Markov process definition earlier stated, whereby history is not necessary. An important example of both a Markov process and a martingale is Brownian motion, which plays a key role in stochastic calculus as well as in mathematical finance.

Definition 3.2.4.2

A portfolio consisting riskless and risky asset is a pair $\phi = (H_t^0, H_t)$ of stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\ t\in T}, P)$ where H_t^0 and H_t are respectively the sizes of risky and riskless asset in possession by an investor at time $t \in [0, T]$. The general worth of this portfolio at a specific time t is

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t$$

We express the essential nature of strategies in which the choice made on the proportion of different parts of the portfolio does not change its worth, in other words, alterations in the portfolio worth would only be the change on the asset value no external withdrawals or additions to the portfolio are made.

Definition 3.2.4.3 A self-financing tamed strategy is a pair ϕ of adapted process $(H_t^0)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ satisfying

1. $$\int_0^T |H_t^0| dt + \int_0^T (H_t)^2 dt < \infty$$
   a.s.

2. $$H_t^0 S_t^0 + H_t S_t = H_t^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u$$
   a.s. $\forall,\ t \in [0, T]$

Definition 3.2.4.4

A strategy $\phi = ((H_t^0, H_t))_{0\leq t\leq T}$ is admissible assuming it is self-financing and supposing the discounted wealth process $V_t(\phi) = H_t^0 S_t^0 + H_s S_t$ of the similar portfolio is positive, for all t, and such that $\sup_{t\in[0,T]V_t}$ is square integrable.

Proposition 3.2.4.1

Let $\phi = (\phi)_{0\leq t\leq T} = (H_t^0, H_t)$ be an adapted process with values in $\mathbb{R}^2$, satisfying

$$\int_0^T |H_t^0| dt + \int_0^T (H_t)^2 dt < \infty$$

a.s. Let $V_t(\phi) = H_t^0 S_t^0 + H_t S_t$ and $V_t(\phi)$. Then ϕ defines a self-financing strategy if and only if

$$V_t(\phi) = V_0(\phi) + \int_0^t H_u dS_u$$

a.s. for all $t \in [0, T]$.

Remark 3.2.4.1

The subject we are dealing with implies that the total alterations in the portfolio make up are done without cost. In pricing option, we require self-funding strategies replicating the option. Firstly, we construct an equal probability measure under which the discounted price of assets are martingale.

3.2.5 Some notions of financial market

Financial market is where diverse people and organisations transact financial securities, commodities, and different items of worth that are exchangeable at affordable cost and price that agree with the principle of supply and demand.

Securities involves stock and Banks, while commodities equally involves metals of higher worth and produce of agriculture.

Economically, a market denotes the group of willing and determined buyers and sellers of items or services and the tradings between them Copland (1992). In an economy however, a security market is used to generate money, transfer real assets in financial assets, ascertain prices that stabilise demand and supply as well as create an environment of investment, a conglomeration of short and long term investments, Brigham and Houston (2007), Brigham and Ehrhardt (2008).
Consequently, financial market is a market that facilitates fund generation or assets investment. It also entails handing of several risks. It can be divided into the following subgroups:

1. Capital market
2. Money market
3. Insurance market
4. Foreign exchange market
5. Derivative market

6. Commodity market

Assets are financial products that are traded in the market. While the economist categorically carry out research for some companies having share price, the financial mathematician study the share price and uses vital mathematical tools such as stochastic calculus to achieve a fair value of derivatives on stock Carr *et al.* (2002) and Cont *2001.*

We further give vital notions of financial markets where assets are traded on a continuous basis to a fixed time horizon T. A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t \in T, P)$ that is complete is our market model. where Ω is the set of scenarios, the set of the parts of Ω is the set of events that may occur while P measures their occurrence.

Financial products traded in the market is an asset. Some of the basic assets or underlying assets traded in the market are:

1. A stock is the contract that permit its owner certain privileges of receiving future dividends and final liquidation price, however, the amount of these payments at execution time is unpredictable Fama (1965).

   To acquire stock, means an investor automatically owns part of a company's assets and earnings French (1980). A shareholder purchase stock with the motive of receiving a compensation in either dividends or capital gains form. The stocks price is presumed to be log-normally distributed and thus can be a minimum zero and a maximum of infinity.

2. Bond: Its a contract that pays its owner a known amount, known as the face value at a known future date. A bond can as well periodically pay its owner fixed cash dividends known as coupons or called a zero-coupon bond otherwise.

3. Commodities: These are physical objects, typically natural resources or goods such as:

1. soft commodities: cocoa,coffee and sugar

2. grains and oil-seeds: barley,corn,cotton,palm Oil etc.

3. metals: copper, nickel,tin,etc.
4. precious metal: gold, platinum,silver, etc
5. livestock: cattle, etc.
6. Energy: crude Oil, fuel Oil,etc.
7. Index(e.g. S and P 500;FT-SEE 100) Simon (2001).
8. Interest rate (e.g. LIBOR).

Another important component of interest in the financial market are contingent claims or derivatives since their evolution solely relies on the primary assets of the market evolution Simon (2001).

Definition 3.2.6.1

A contingent claim with maturity $T > 0$ is an arbitrary $\mathcal{F}_t$-measurable random variable X.

Remark 3.2.6.1

A financial contract whose value at expiration depends on the price(s) of assets underlying is a contingent claim. Some examples of contingent claims are:

1. Forwards and Futures: This is an agreement binding two counterparts, in which one agrees to buy an asset from the other at some specified time in the future, referred to as the delivery date, for some specified price.

   The terms of the market contract necessitate the need to purchase the asset at the delivery date. The amount paid for the asset is referred to as the delivery price and is set when the contract is entered into. The main difference between forward and futures contracts is that futures are traded through an exchange whereby the parties do not interact directly and counterpart risk, the risk that either party will default, is assumed by the exchange.

   Forwards are traded in what is referred to as the over-the-counter(OTC)market, whereby counterparts interact directly with one another Stout (1999). OTC market is non-regulated and contracts are non-standardized,there is always the possibility of a credit-risk. The foreign exchange movements are effectively hedge with forwards.

2. Swaps: A swap is a contract binding two investors in agreement to make exchange in the future for several financial assets(either cash-flows) at a known date based on the agreed formula that rely on the underlying assets value. For instance accuracy swaps(exchange currencies) and interest rate swaps(exchange fixed for floating interest payments).

3. Options: These are particular derivatives, characterized by non-negative payoffs. That is, option provides its owners with the possibility of endless profit at a limited loss risk Gerber (1994) and Co *et al.* (2000). At a fundamental level option is partition into two types: such as call options and put options. Options are the most common types of derivatives Britten *et al.* (2000).

   However, options are of two types: standard options, also known as vanillas, there are actively traded at the exchange; as well as exotic options, which are meant to satisfy the needs of their writers clients. There do not have active market, therefore their value is achieved by using a model to determine the premium.

Definition 3.2.6.2

An option is a financial contract that grants its holder the right, without any obligation to buy (or sell) a specific underlying asset on or before a given date $T \geq 0$ in the future (expiry date) for an agreed price $K \geq 0$, called the exercise or strike price. Accordingly, one distinguishes between a call option (the right to buy) and a put option (the right to sell), and between an European option (exercise only at time T)and an American option(exercise at time $t \in [0, T]$).

The time-dependent stock price underlying the option is denoted by $S = S(t) = S_t$ for $t \geq 0$ and is assumed to yield dividend. Usually, underlying asset is not transferable at the exercised of option, rather a cash flow of the money achieved. The more flexible the option, the more expensive it will be. there are options not on the underlying assets, but also on other derivatives. The range of possible options is essentially unlimited.

3.2.6 Some examples of options

1. European Option: which may be exercised by its holder only at maturity date T.

2. American Option: which may be exercised by its holder any time up to maturity date $t \leq T$. An American derivative type is a contract whose cash flows is usually influenced by the owner of the derivative. The owner influences the flow of cash through the exercise strategy.

3. Bermudan Option: which may be exercised at maturity or at specific dates before maturity. Bermudan option is intermediate between a European option and an American option.

4. Asian Option: The payments promised by this option depend on the average primitive asset price. They are useful tools for hedging those volatile assets. Several averages are usually used in the Asian type of contracts e.g. arithmetic, geometric, weighted-arithmetic, e.t.c.

5. Israeli Option: An Israeli option is also a plain vanilla option which consists some features that enable the seller of the option terminate the contract but must pay the early exercise payoff and the penalty fee.

6. Russian Option: This pays the owner of the option the maximum price gained by the underlying asset over the life of the option, discounted at some rate $\lambda > 0$ Scheikman (2003). It is also called reduced regret

7. Barrier Option: This is a path-dependent option where payoff rely on whether the underlying asset price hits a specific value during the option's lifetime.

$$h = (X_T - K)^+_{1_{\inf X_t > B}}$$

8. Basket Option: The payoff of the option depends on the values of more than one risky underlying asset, which typically do not evolve independently. The payoff function of the basket option on underlying assets S^i is given by $h = (\sum w_i s^i - K)^(+)$, where w_i denotes weighted or share of the $i^t h$ underlying asset in basket.

9. Path-dependent Option: This is an option whose payoff do not only rely on the stock price value at maturity but equally on the past history of the underlying asset price.

10. Lookback Option: These are path-dependent options whose payoff rely on the max or min of the underlying asset price attained over a certain period of time known as the lookback period. lookback option can be broadly classified into: fixed strike and floating strike.

11. Perpetual option: A perpetual option is an American option with infinite time horizon. For pricing perpetual options in B-S model, we only need to modify the B-S model by removing the time-derivative term to obtain a steady-state equation.

12. Passport option: It is call option on the trading account, that is, the owner receives the positive part of the value of his trading account.

13. The Rainbow option: It involved several risky underlying. It is equally synonymous to the basket option rather its underlying assets weight rely on their performance. It pays a fixed amount or nothing at all, depending on the price of the underlying assets at maturity. They have discontinuous payoffs.

14. Shout option: A shout option entitles the holder to alter certain features of the contract during the life of the contract according to some pre-specified conditions. the time of modification, known as the reset time, may be chosen optimally by the holder or may be automatic upon fulfillment of some certain preset requirements. In practice the terms which may be reset are the strike and maturity of the option.

15. Tokyo option: This is also referred to as knockout barrier option consisting region of caution. The option usually moves further from the safety position into the caution status after the barrier first hit. But if the barrier hit while still in the caution state, the option vanishes Fujita and Minura, (2002).

16. Spread option: The payoff of this option rely on the difference of two underlying assets. They are typically with multi-asset option and are widely used in the commodity markets.

17. Quanto option: This option is a financial derivative comprised of underlying assets in one currency and payoff in another one. Options are not only on

underlying assets, but also on other derivatives. Options are used for several purposes. The two most important are: speculating and hedging. Participants in the option market are: hedgers, speculators and arbitrageurs.

18. Hedgers: Often the future fluctuations of underlying assets price is not known in advance today. Therefore, traders are more careful to avoid future risk occurrence. These traders are hedgers and this lofty strategy is known as hedging.

19. Speculators: Since the intention of hedgers are to avoid risk, speculators are traders in the financial market with the motive of making profit. If a speculator is not sure of profit, he forfeit the option.

20. Arbitrageurs: The inefficiencies of financial market existence are unfortunately taking undue advantage off by these set of traders to make riskless profit through this opportunities. Depending on when options are exercised, they are classified into three groups: European, Bermudan and American options Glasserman,(2003).

21. Pricing: this explains option from the perspective of the owner; it based on the problem of finding if it does exist, the fair price of a derivative at any time $t < T$. However, an important question is about the existence of a deterministic function given the value of the derivative.

22. Hedging: This means the writer will minimise the risk related to the derivative he wish to transact and seeks for a deterministic hedging strategy. To proffer a reasonable solution to both problems, it is necessary to find a suitable model for the market which suit the data of the underlying assets. Rebonato (2004), Renault (1996) and Rubinstein (1994). The basic assumptions below are necessary for modelling procedure:

1. trading takes place continuously in time

2. the riskless interest rate r is known and constant over time.

3. the assets are perfectly divisible i.e. part of the assets can be trade on

4. Riskless arbitrage opportunities are excluded

5. the assets take the part of geometric Brownian motion(GBM) Trading in financial market involves the use of a portfolio of assets to formulate a hedging strategy

Definition 3.2.7.1

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A probability measure Q on $(\Omega, \mathcal{F})$ is totally continuous with respect to P if $\forall\ A \in \mathcal{F}\ P(A) = 0 \rightarrow\ \ Q(A) = 0$. Delbaen (1995)

Theorem 3.2.7.1

Q is absolutely continuous in relation with P supposing and assuming there exists a positive random variable Z on $(\omega, \mathcal{A})$ such that $\forall A \in \mathcal{A}$, $Q(A) = \int_{\mathcal{A}} Z(w)dP(w)$. Z is called density of Q Relative to P and represented as $\frac{dQ}{dP}$

Definition 3.2.7.2

Let Q and P be two probability measures on $(\omega, \mathcal{A})$, while P and Q are equal assuming each one of them is absolutely uniform with respect to the other. Furthermore, a probability measure Q is similar with a given probability measure P as explained by the theorem below

Girsanov's Theorem (1960)

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t \geq 0, p)$ be a probability space with $(\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of the standard Brownian motion $(B_t)_{0 \leq t \leq T}$. Let $(\phi_t)_{0 \leq t \leq T}$ be an adapted process satisfying the equation below $\int_0^T \phi_s^2 ds < +\infty$ a.s. in the sense that the equation $(L_t)_{0 \leq t \leq T}$ expressed as $L_t = \exp(-\int_0^t \phi_s dB_s - \frac{1}{2}\int_0^t \phi_s^2 ds)$ is a martingale. Then, with the probability $P^{(L)}$ and density L_T relative to P, the process $(W_t)_{0 \leq t \leq T}$ defined by $W_t = B_t + \int_0^t \phi_s ds$ is a Brownian motion. As a result, the stochastic integral does not vary even with the change of measure of similar probability.

Proposition 3.2.7.1

If the hypothesis of our previous Theorem are meet, then set $(H_t)_{0 \leq t \leq T}$ to be an adapted process such that $\int_0^T H_s^2 ds < \infty$ P a.s. Let the processes

$$X_t = \int_0^t H_s dB_s + \int_0^t H_s \phi_s ds$$

under P and

$$Y_t = \int_0^t H_s dW_s$$

under $P^{(L)}$ with $W_t = B_t + \int_0^t \phi_s ds$ and $P^{(L)}$ our earlier stated measure of probability in the previous Theorem. Then $X_t = Y_t$. Our result on the representation of a

Brownian martingale in terms of stochastic integral is next.

Theorem 3.2.7.2 Musiela (2004) and Girsanov (1960)

On the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, set $(B_t)_{0\leq t\leq T}$ to be a standard Brownian motion and $(\mathcal{F}_t)_{0\leq t\leq T}$ its natural filtration. Furthermore, Let $(M_t)_{0\leq t\leq T}$ be a square-integrable martingale, with respect to $(\mathcal{F}_t)_{0\leq t\leq T}$. Then There exists a fitted process $(H_t)_{0\leq t\leq T}$ in the sense that $E(\int_0^T H_s^2 ds) < +\infty$ and

$$\forall t \in [0,T] \;\; M_t = M_0 + \int_0^t H_s dB_s$$

a.s.

Definition 3.2.7.3

An option is replicable if there is an admissible strategy $\phi = ((H_t^0, H_t))_{0\leq t\leq T}$ in such a way that at any time T its worth is equivalent to the option payoff $V_t(\phi) = h$.

Remark 3.2.7.1

For an option to be replicable, h ought to be square integrable with respect to Q. This necessary condition is fulfilled when h is represented as $h = g(S_T)$, with $g(x) = (x-K)^+$. Our next result states the option price.

Theorem 3.2.7.3 Lamberton *et al.* (1996)

In the Black-Scholes model, any option defined by a positive $\mathcal{F}_t$- measurable random variable h, and square-integrable in respect of the probability measure Q, can be replicated and the worth at time t of any replicating portfolio is represented as $V_t = E_Q(e^{-r(T-t))}h|\mathcal{F}_t$. The expression $E_Q(e^{-r(T-t))}h|\mathcal{F}_t$ defines the option value at time t. If h is square integrable with respect to Q, then there is an acceptable strategy replicating the option.

3.2.7 The Black-Scholes model

The classical option pricing theory of Black-Scholes, (1973) is based on a continuous-time model with one risky asset with price S_t at time t and a riskless asset with price S_t^0 at time t satisfying the ordinary differential equation (ODE)

$$dS_t^0 = rS_t^0, \; t \geq 0,$$

where $r > 0$ is the riskless interest rate. To model the stock price we fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and complete. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on a probability space. With the drift μ and the volatility σ the stock price is determined by the following stochastic differential equation(SDE):

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad t \geq 0$$

Applying Itô's formula yield the explicit representation of the unique solution of the SDE.

$$S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t), t \geq 0$$

The classical option pricing theory is based on portfolio investment replication of any option through the underlying stock and bond. In a complete market, there exists a unique probability measure Q equivalent to the measure P under which the discounted stock price $S_t := \exp(-rt)S_t$ is a martingale Lamberton and Lapeyre, (1996).

3.2.8 Methods of valuation of option

1. An European option stated by a positive $\mathcal{F}_t$-measurable random variable $h(S_T)$, the payoff function can be replicated and the value at time t in Lamberton and Lapeyre, (1996) is given by

   $$V(S_t, t) = E_Q(e^{-r(T-t)h(S_T)|\mathcal{F}_t)}$$

   where E_Q represent the expectation under Q, an equivalent martingale measure (EMM) to P that is, $Q \sim P$ such that, the discounted process is a Q-martingale. our previous equation is called the right-neutral valuation method of European options.

2. However, the Black-Scholes model is based on the assumption that the asset pays no dividends. Then the price $V = V(S,t)$ of an European option satisfies the backward parabolic PDE.

   $$\frac{\partial V}{dt} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial} S^2 + rS\frac{\partial V}{\partial S} - rV = 0$$

   $$V(S,T) = h(S), \quad (S,t) \in (0,\infty) \times [0,T]$$

   this equation is called the Black-Scholes equation for the valuation of derivative prices.

3. Merton (1973) and (1976) extended the Black-Scholes equation to options on dividend paying shares, given by:

$$\frac{\partial V}{dt} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial} S^2 + (r - d) S \frac{\partial V}{\partial S} - rV = 0$$

$$V(S,T) = h(S), \quad (S,t) \in (0,\infty) \times [0,T]$$

where S is the spot price of the underlying asset, T is the maturity date, r is the risk-free interest rate and d is the dividend yields.

3.3 Utility function

Utility function is the measure of the well being of a consumer for all combinations of goods consumed and their welfare as well as their preferences over some set of goods and services.

The concept is important in financial mathematics, and the rational theory of choice in economics and games theory, because it represents the satisfaction derived by the consumer of a good. However, a good is something that satisfy human wants Mark Schroder (1999). some examples of utility functions are:

1. Logarithmic utility function: $\eta(x) = \log(x)$
2. Power utility function: $\eta(x) = \frac{x^c}{c}$, $c \leq 1$
3. exponential utility function: $\eta(x) = 1 - \exp^{-\alpha x}$
4. Quadratic utility function: $\eta(x) = x - \beta x^2$

The logarithmic utility is the limit of $\frac{x^{(1-c)}}{c}$ as c tends to zero. The utility $\frac{x^{(1-c)}}{c}$ is equivalent to the power utility, such that they generate similar preferences.

Logarithmic utility function is not in the family of utility functions with consistent relative risk aversion.

In power utility, the parameter c is the risk- aversion. $\frac{x^{(1-c)}}{c}$ for $c > 0$ indicating the investors risk tolerance. The logarithmic utility corresponds to a risk-neutral investor. In this case, η'' is zero.

Example 3.3.1

For instance we have $\eta(x) = \ln(x)$ the first derivative of that equation is $\eta'(x) = \frac{1}{x}$ and the second derivative is $\eta''(x) = -\frac{1}{x^2} < 0$ for the interval $0 < x < \infty$ However, if $\eta(x) = (\ln x)^2$ then $\eta'(x) = \frac{2\ln x}{x}$ and $\eta''(x) = \frac{2(1-\ln x)}{x^2}$ this function is not concave on its whole domain. It is concave only for x such that $\ln x \geq 1$,

Example 3.3.2

Let $\eta(x) = 1 - \exp^{-bx}$ where $b > 0$, then $\eta'(x) = b\exp^{-bx}$ and $\eta''(x) = -b^2\exp^{-bx}$. Since $b > 0$ then $-b^2 < 0$ and thus $\eta''(x) \leq 0$ for all x Hence $\eta(x)$ is concave. The function $\eta(x)$ is monotone increasing since $\eta'(x) > 0$ for all x. Therefore if $0 \leq x_1 < x_2$ then $\eta(x_1) < \eta(x_2)$.

A good investor is subject to rational decision making for instance, supposing an investor is faced with the choice of two different investment products and assumed further that the set of outcomes resulting from investing in either of the products is $\{e_1, e-2, \cdots e_n\}$.

We set the probability of the first product of investment outcome as P_i for $\{i = 1, 2, \cdots, n\}$. Supposing the outcome e_i did not emerge from first product investment, then $P_i = 0$. In the same vain we represent the value of the second products probability by q_i.

The investor decides ranking the outcomes in order of desirability, with the most desirable outcome as e_α and the least as e_μ. For either of the possible outcomes, a utility is assigned.

The utility function $\eta(e_i)$ range is $[0, 1]$. That is, $\eta(e_\alpha) = 1$ as well as $\eta(e_\mu) = 0$. The values of $\eta(e_i)$ for which $i \neq \alpha,\ \mu$ lies in the open interval $(0, 1)$. Assigned as follows.

He might reason that between the choice of receiving the outcome e_i with assurance or be involved in a random experiment and receive the most desirable outcome e_α with the probability η and the least desirable outcome e_μ with $1-\eta$ probability.

Assuming $\eta = 1$, it means the investor will definitely undertake the random experiment to receiving the certainty e_α, which they consider more desirable in the stead of e_i. Supposing $\eta = 0$, this means that the investor will not participate since its results gives the least desirable outcome. They would prefer the sure outcome e_i.

Furthermore, as η reduces from 1 to 0 there will be a value of $0 < \eta < 1$ in which he changes his mind from taking part in the random experiment to undertake the outcome e_i with assurance. In this case, η is $\eta(e_i)$. This probability is known as the utility of the outcome e_i.

This particular point of probability in which the investors behaviour or the mind of an investor changes is unique. Then a cognitive investor that initially accepted the sure outcome e_i can not at a lesser probability of receiving e_α decide to undertake in the random experiment.

Therefore, two outcomes of unequal desirability are compared with utility function. The outcome e_i is synonymous to receiving e_α with the $\eta(e_i)$ probability or receiving e_μ with $1 - \eta(e_i)$ probability. The decision between two different investment, if the investment on the first product is similar to receiving e_α with probability $\sum_{i=1}^{n} P_i\eta(e_i)$ while the second investment is synonymous to receiving e_α with the probability $\sum_{i=1}^{n} q_i\eta(e_i)$. The investor chooses the first product whenever he observed that

$$\sum_{i=1}^{n} P_i\eta(e_i) > \sum_{i=1}^{n} q_i\eta(e_i)$$

or chooses the second if otherwise.

So far, we have discuss the outcomes or consequences e_i of choosing investment decision. Consequently, these outcomes is seen as receiving different amount of money for an investment of which most of them is non-positive.

Therefore, the utility function represented by $\eta(x)$ is the investor's utility of receiving an amount x. Utility functions are like personality traits to each investor.

Next, we give some vital properties of utility function. One property of utility function is that the amount of extra utility received by an investor when x increased to $x + \Delta x$ does not increase. that is, $\eta(x + \Delta x) - \eta(x)$ is a non-increasing function of x. It is also concave on an open interval (a, b) if for every $x, y \in (a, b)$ and for every $\lambda \in [0, 1]$, we have

$$h(\lambda x + (1 - \lambda)y \geq \lambda h(x) + (1 - \lambda)hy)$$

An investor whose utility function is concave is said to be risk-averse, while an investor with a linear utility function of the form $\eta(x) = ax + b$ with $a > 0$ is known as risk-neutral.

Definition 3.3.1

The value of a utility function is known as expected utility. In our earlier discussion on choosing between two possible investment, an investor with the greater utility is preferable.

$$\sum_{i=1}^{n} P_i \eta(e_i) > \sum_{i=1}^{n} q_i \eta(e_i)$$

The above equation highlighted that supposing the investment choice results in outcomes $\{e_1, e_2, \cdots e_n\}$ with respective probabilities P_i for $i = 1, 2, \cdots, n$ and the second investment choice produces a similar outcomes with q_i for $i = 1, 2, \cdots, n$ probabilities then a sensitive investor pick the first investment whenever

$$\sum_{i=1}^{n} P_i \eta(e_i) > \sum_{i=1}^{n} q_i \eta(e_i)$$

. Assuming X is the set of outcomes of a random variable with probabilities P_i for $i = 1, 2, \cdots, n$, Y represents the outcomes with q_i for $i = 1, 2, \cdots, n$, thus the first investment is preferable whenever

$$E[\eta(X)] > E[\eta Y]$$

3.4 Forward Integral

This section discusses the concept and application of forward integral equations to the optimal portfolio problem of a sensitive investor. For instance, we assume risky assets price at a specific time τ is $D(\tau)$ and also assume that an investor acquires a stochastic amount α of valuables at unpredictable periods τ_1, and hold it till a random time $\tau : \tau_1 < \tau_2 < \xi$, and eventually puts it on sale. The profit obtained is

$$\alpha D(\tau_2) - \alpha D(\tau_1)$$

This expresses an exact scenario of the achieved result with a portfolio driven by forward integration.

$$\varphi(\tau) = \alpha I(\tau_1, \tau_2](\tau), \ \tau \in [0, \xi],$$

in relation with the Brownian motion, i.e.,

$$\int_0^{\xi} \varphi(\tau) d^- D(\tau) = \alpha D(\tau_2) - \alpha D(\tau_1).$$

Assuming in a financial market on $[0, \xi]$ $(\tau > 0)$ between capital placement of two opportunities:

(1) An asset that is riskless per unit of price $L_0(\tau)$ at τ given by

$$d\text{Ł}_0(\tau) = \lambda \text{Ł}_0(\tau)d\tau, \ \ \text{Ł}_0(0) = 1,$$

(2) A risky asset with unit price $\text{Ł}_1(\tau)$ at time τ given by

$$dL_1(\tau) = \text{Ł}_1(\tau)\left[\vartheta d\tau + \omega dD(\tau)\right], \ \ \text{Ł}_1(0) > 0.$$

λ, ϑ, and $\omega > 0$ are constant.Let

$\nu(\tau)$ be a portfolio and fraction of money $L(\tau)$ invested in the risky at τ, then the evolution of the wealth process $L_\nu(\tau) = L(\tau), \ \tau \geq 0$, of a self-funding portfolio ν is

$$\begin{aligned} dL(\tau) &= (1-\nu(\tau))L(\tau)\lambda d\tau \\ &+\nu(t)L(\tau)[\vartheta d\tau + \omega dD(\tau)] \\ = \text{Ł}(\tau)&\left[\lambda + (\vartheta-\lambda)\nu(\tau)d\tau + \nu(\tau)\omega dD(\tau)\right], \\ L(0) &= s > 0 \end{aligned} \tag{3.1}$$

ν is the portfolio of all $\mathbb{F}$-adapted set $\mathcal{H}_\nu$ such that

$$\int_0^\xi \nu^2(\tau)d\tau < \infty \quad \text{p.a.s.}$$

Assuming $\nu \in \mathcal{H}_\nu$, then the result $L(\tau) = L^s_{(\tau)}, \ \tau \in [0,\xi]$, of (3.1) leads us to the above equation

$$L(\tau) = s\exp\left\{\int_0^\tau \omega\nu(s)dD(\tau) + \int_0^\tau [\lambda + (\vartheta+\lambda)\nu(s)\frac{1}{2}\omega^2\nu^2(s)]ds\right\} \tag{3.2}$$

Then the portfolio ν^*_ν that guarantees highest profit

$$E[\log L^s_\nu(\xi)] \quad (\xi > 0) \tag{3.3}$$

over all $\nu \in \mathcal{H}_\mathbb{F}$ is given by

$$\nu^*_\mathbb{F}(s) = \frac{\vartheta-\lambda}{\omega^2}, \ s \in [0,\xi] \tag{3.4}$$

then the value function is:

$$\begin{aligned} V^{(s)}_\nu &= \sup_{\nu\in\mathcal{H}_\mathbb{F}} E[\log L^s_\nu(\xi)] \\ &= E[\log L^s_{\nu^*_\mathbb{F}}(\xi)] \\ = \log s &+ \left(\lambda + \frac{(\vartheta-\lambda)^2}{2\omega^2}\right)\xi, \ s > 0 \end{aligned} \tag{3.5}$$

But, an investor could be receptive to possessing another information other than $\mathcal{F}_{\bar{s}}$? specifically, if $\mathbb{G} = \{\mathcal{G}_s,\ s \geq 0\}$ is one of such filtration in the sense that

$$\mathcal{F}_s \subseteq \mathcal{G}_s,\ s \in [0, \xi] \quad (\xi > 0),$$

$$\mathcal{G}_s := \mathcal{F}_s \vee \sigma(B(\xi_0)) \quad (\xi_0 > \xi) \tag{3.6}$$

Which implies that the investor is aware of the value of $B(\xi_0)$ which is also similar to knowing $L_1(\xi_0))$ in addition to the initial information $\mathcal{F}_t$ at any time $s \in [0, \xi]$. This is to aid a sensitive investor hedge against any risk by diversifying.

Definition 3.4.1

A measurable stochastic process $\varpi = \varpi(s), \quad s \in [0, \xi]$, is forward integrable in relation with the Brownian motion assuming $I(s), \quad s \in [0, \xi]$, exists in the sense that

$$\sup_{s \in [0,\xi]} \left| \int_0^t \varpi(s) \frac{B(s+\varepsilon) - B(s)}{\varepsilon} ds - I(t) \right| \to 0, \ \ \varepsilon \to 0 \tag{3.7}$$

in probability. Then, for any $s \in [0, \xi]$,

$$I(s) = \int_0^s \varpi(s) d^- B(s)$$

is forward integrable of ϖ relating to the Brownian motion $B(s)$ on $[0, \xi]$.

Lemma 3.4.1

Assuming ϖ holds as stated in Definition (3.4.1) then

$$\int_0^\xi \varpi(s) d^- B(s) = \lim_{\Delta t \to 0} \sum_{j=1}^{J_n} \varpi(s_j - 1)(B(j_1) - B(s_j - 1))$$

a likelihood intersection. Taking limit within the partitions of $0 = \tau_0 < \tau_1 < \cdots \tau_{J_n} = \xi$ of $\tau \in [0, \xi]$ with $\Delta_s := \max\ j = 1, \cdots, J_n(\tau_j - \tau_{j-1}) \to 0$ as $n \to \infty$.

Proof

set ϖ as

$$\varpi(s) = \sum_{j=1}^{J_n} (s_j - 1) \chi(s_{j-1}, s_l](s), \quad t \in [0, \xi],$$

a simple stochastic process, however, a Cáglád process ϖ on $s \in [0, \xi]$ can equally be estimated point-wise and consistently in B .

then

$$
\begin{aligned}
\int_0^{\xi} \varpi(s) d^- B(s \quad) &= \lim_{\varepsilon \to 0^+} \int_0^{\xi} \varphi(s) \frac{B(s+\varepsilon) - B(s)}{\varepsilon} ds \\
&= \sum_{j=1}^{J_n} \varpi(s_j - 1) \lim_{\varepsilon \to 0^+} \int_{s_{j-1}}^{s_l} \frac{B(s+\varepsilon) - B(s)}{\varepsilon} ds \\
&= \sum_{l=1}^{J_n} \varpi(s_j - 1) \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{s_{j-1}}^{s_j} \int_{s}^{s+\varepsilon} dB(u) ds \\
&= \sum_{j=1}^{J_n} \varphi(s_j - 1) \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{s_{j-1}}^{s_l} \int_{u-\varepsilon}^{u} ds \, dB(u) \\
&= \sum_{j=1}^{J_n} \varpi(s_j - 1)(B(s_j) - B(s_{j-1})),
\end{aligned}
$$

Remark 3.4.1

When ϖ is $\mathcal{F}$-adapted, Riemann sum are roughly Itô integral of ϖ with respect to $\tilde{B}$. Consequently, a non-anticipating form of the forward integral is also an extension of the Itô integral. However, effort has been made to extend this concept to a sensitive investors settings. We presents this in lemma 3.4.2

Lemma 3.4.2

furthermore, supposing $D(\tau)$ is $\mathcal{N}_\tau$ -semimartingale and $\varpi(\tau)$ is $\mathcal{N}_\tau$-measurable on $\tau \in [0, \xi]$. Then

$$
\int_0^{\xi} \varpi(\tau) dD(\tau) = \int_0^{\xi} \varpi(\tau) d^- D(\tau),
$$

Proof

Since D is a semimartingale with respect to $\mathbb{N}$ and ϖ is $\mathbb{N}$-predictable, by Itô Proske, F. (2007) we have

$$
\begin{aligned}
\int_0^{\xi} \varpi(\tau) d^- D(\tau) &= \lim_{\rho \to 0^+} \int_0^{\xi} \varpi(\tau) \frac{D(\tau+\rho) - D(\tau)}{\rho} d\tau \\
&= \lim_{\rho \to 0^+} \int_0^{\xi} \frac{1}{\rho} \int_{\tau-\rho}^{\tau} \varpi(\tau) d(\tau) dD(\tau) \\
&= \int_0^{\xi} \varpi(\tau) dD(\tau),
\end{aligned}
$$

since

$$\varpi\rho(\tau) := \frac{1}{\rho}\int_{\tau-\rho}^{\tau} \varpi(\tau)d\tau,\ \tau \in [0, \xi],$$

approaching ϖ in probability and consistently in τ.

$\square$

3.4.1 Itô formula for forward Integral in One Dimensional Case

We now discuss Itô formula for forward integral. In this context, it is vital to introduce a notation that is synonymous to the classical notation for Itô process.

Definition 3.4.2

Let D be a forward stochastic process and further expressed as

$$L(\tau) = x + \int_0^\tau \varphi(s)ds + \int_0^\tau v(s)d^- D(s), \tau \in [0, \xi] \tag{3.8}$$

(x constant), where

$$\int_0^\xi |\varphi(\tau)|d\tau < \infty, p - \hat{a}.\hat{s}. \tag{3.9}$$

and $\tilde{v}$ is a forward stochastic process. the short form of (3.9) is

$$d^- L(\tau) = \varphi(\tau)d\tau + v(\tau)d^- D(\tau) \tag{3.10}$$

Theorem 3.4.1 Proske, F. (2007)

(The one-dimensional Itô formula for forward integrals). Now let

$$d^- L(t) = \varphi(t)dt + v(t)d^- D(t)$$

be a forward process. We set $\varphi \in C^{1,2}([0, \xi] \times \mathbb{R})$ and define

$$\varsigma(t) = \varphi(t, L(t)), \quad t \in [0, \xi].$$

Then $\varsigma(t),\ \ t \in [0, \xi]$, is a forward process and

$$d^- \varsigma(t) = \frac{\partial \varphi}{\partial t}(t, L(t))dt + \frac{\partial \varphi}{\partial L}(t, L(t))d^- L(t) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial L^2}(t, L(t))v^2(t)dt$$

Theorem 3.4.2 Proske, F. (2007)

(The multidimensional Itô formula for forward integrals). Let

$$d^- L_i(t) = \varphi(t)dt + \sum_{j=1}^{m} V_{ij}(t)d^- D_j(t) \quad (i = 1, \cdots, n)$$

be n forward processes driven by m independent Brownian motions.

$$D_1, \cdots, D_n.$$

Let $f \in C^{1,2}([0,\xi] \times \mathbb{R}^n)$ and define $\varsigma(t) := f(t, L(t)), \quad t \in [0,\xi]$.

ς is a forward process and

$$\begin{aligned} d^-\varsigma(t) &= \frac{\partial f}{\partial t}(t, L(t))dt + \sum_{i=1}^{n} \frac{\partial f}{\partial L_1}(t, L(t))d^- L_i(t) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial L_i \partial L_k}(t, L(t))(V^\xi)_{ik}(t)dt. \end{aligned}$$

As in the classical case, we can use Itô formula for forward integrals to solve forward stochastic differential equations. We show this by an example.

Example 3.4.1

Let $\vartheta(\tau)$, and $\omega(\tau)$, $\tau \in [0,\xi]$ be measurable process, then

(1) ω is forward integrable.

(2) p-a.e. $\int_0^\xi (|\vartheta(\tau)| + \omega^2(\tau))d\tau < \infty$

$\tau \in [0,\xi]$ $\varsigma(\tau)$, is the unique solution of

$$\begin{aligned} d^-\varsigma(\tau) = \vartheta(\tau)\varsigma(\tau)d\tau \\ +\omega(\tau)\varsigma(\tau)d^- D(\tau), \tau \in [0,\xi] \\ \varsigma(0) = F \end{aligned} \tag{3.11}$$

where F is a random variable of $\mathcal{F}_\tau$-measurable, then

$$\varsigma(\tau) = F \exp\left\{\int_0^\tau (\vartheta(s) - \frac{1}{2}\omega^2(s) + \int_0^\tau \omega(s)d^- D(s)\right\}.$$

Applying Itô forward integrals concept for $F(x) = e^x$, $x \in \mathbb{R}$, then

$$dL(\tau) = (\vartheta(\tau) - \frac{1}{2}\omega^2(\tau))d\tau + \omega(\tau)d^- D(\tau) \tag{3.12}$$

ς is a unique solution, assume ς is another solution. Then Consider

$$\upsilon(s) := \varsigma^{-1}(s)\varsigma(s)s \in [0,\xi] \tag{3.13}$$

then by the two-dimensional It $\hat{o}$ formula for forward integral processes, we have

$$\begin{aligned} d^-\upsilon(s) &= \varsigma(t)d^-\varsigma^{-1}(s) + \varsigma^{-1}(s)d^-\varsigma(s) + d^-\varsigma^{-1}(s)d^-\varsigma(s) \\ &= \varsigma^{-1}(s)\varsigma(s)\left[(-\vartheta(s) + \omega^2(s))ds - \omega(s)d^- D(s)\right. \\ &+ \left.\vartheta(s)ds + \omega(s)d^- D(s) + (-\omega(s))\omega(s)ds\right] = 0. \end{aligned}$$

Since $\varsigma(0) = \varsigma(0) = F$, it follows that $\upsilon(s) = 1, s \in [0,\xi]$, which proves uniqueness.

Chapter 4
RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we discuss some important applications of forward integral and its relation with a sensitive investor in a financial market.

4.2 Specification of optimal portfolio of small scale investors

In this section, we discuss how a small scale investor such as a low or average income earner can also maximise portfolio. Consider two securities in a financial market:

1. A non-risky asset (e.g., a Bank account), where the price $L_0(\tau), \tau \in [0,\xi]$ per unit at time (τ) is given by the differential equation

$$dL_0(\tau) = \lambda L_0(\tau)d\tau,\ L_0(0) = 1$$

   and

2. A risky assets (e.g., a stock), where the price $L_1(\tau),\ \tau \in [0,\xi]$ per unit at time τ is given by the stochastic differential equation

$$dL_1(\tau) = \vartheta(\tau)L_1(\tau)d\tau + \omega(\tau)L_1(\tau)d^-D(\tau)\ L_1(0) > 0,$$

$\lambda(\tau), \vartheta(\tau)$ are constant coefficients and $\omega(\tau), \tau \in [0,\xi]$ are deterministic functions on assumption that

$$E[\{\int_0^\xi (|\lambda(\tau)| + |\vartheta(\tau)| + \omega^2(\tau))\}dt < \infty,\ p - a.s.$$

where $\omega(\tau)$ is *cáglád*.

Let $(\nu_0(\tau), \nu_1(\tau)),\ \tau \in [0,\xi]$ denotes portfolios. Then, its value at time τ is represented as

$$X(\tau) = \nu_0(\tau)L_0(\tau) + \nu_1(\tau)L_1(\tau).$$

However, it is self financing assuming

$$dX(\tau) = \nu_0(\tau)dL_0(\tau) + \nu_1(\tau)dL_1(\tau).$$

From

$$X(\tau) = \nu_0(\tau)L_0(\tau) + \nu_1(\tau)L_1(\tau).$$

we make $\nu_0(\tau)$ the subject

$$\nu_0(\tau) = \frac{X(\tau) - \nu_1(\tau)L_1(\tau)}{L_0(\tau)}$$

and substituting in our self financing equation of

$$dX(\tau) = \nu_0(\tau)dL_0(\tau) + \nu_1(\tau)dL_1(\tau)$$

and using

$$dL_0(\tau) = \lambda L_0(\tau)d\tau,\ L_0 = 1$$

and

$$dL_1(\tau) = \vartheta(\tau)L_1(\tau)d\tau + \omega(\tau)L_1(\tau)dD(\tau),\ L_1(0) > 0$$

then

$$dX(\tau) = (X(\tau) - \nu_1(\tau)L_1(\tau))\frac{dL_0(\tau)}{L_0(\tau)} + \nu_1(\tau)dL_1(\tau)$$
$$= \lambda(\tau)X(\tau)d\tau + \nu_1(\tau)L_1(\tau)(\vartheta(\tau) - \lambda(\tau))d\tau + \omega(\tau)dD(\tau)$$

since $\omega(\tau) \neq 0$ for $a.a.$ written as

$$dX(\tau) = \lambda(\tau)X(\tau)d\tau + \omega(\tau)\nu_1(\tau)L_1(\tau)(\alpha(\tau)d\tau + dD(\tau))$$

where $\alpha = \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}$ Set $\nu_1(\tau) = \beta_1\gamma_1$ the part payment or fraction of money invested on stock. Then a sensitive investor have desirable return when he diversify his investment on portfolio that is, where $\omega(\tau)\,(\beta_1(\tau)\gamma_1(\tau))$ is cáglád, $\mathcal{N}_\tau$-adapted and forward integrable stochastic process such that

$$\int_0^\xi \{|\vartheta(\tau)| - r(\tau)|\beta_1(\tau)\gamma_1(\tau)| + \omega^2(\tau)\beta_1^2(\tau)\gamma_1^2(\tau)\}d\tau < \infty$$

a.s. holds, then the wealth

$$X(\tau) = X_{\beta_1(\tau)\gamma_1(\tau)}$$

of a sensitive investor at τ would satisfy

$$d^{-}X(\tau) = X(\tau)[\{\lambda(\tau) + (\vartheta(\tau) - \lambda(\tau))\beta_1(\tau)\gamma_1(\tau)\}d\tau \\ +\omega(\tau)\beta_1(\tau)\gamma_1(\tau)d^{-}D(\tau)],\ X_0 = x_0. \tag{4.1}$$

By Itô formula, we have

$$X(\tau) = \exp\int_0^s [\lambda(\tau) + (\vartheta(\tau) - \lambda(\tau))\beta_1(\tau)\gamma_1(\tau) - \frac{1}{2}\omega^2(\tau)\beta_1^2(\tau)\gamma_1^2(\tau)]d\tau \\ +\int_0^s \omega(\tau)\beta_1(\tau)\gamma_1(\tau)d^{-}D(\tau) \tag{4.2}$$

considering $\beta_1^*\gamma_1^* \in \mathcal{A}_{\mathbb{N}}$ in the sense that

$$\sup_{\beta_1\gamma_1\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{\beta_1(\xi)\gamma_1(\xi)})] = H[\eta(X_{\beta_1{}^*(\xi)\gamma_1(\xi)})].$$

We defined the family of admissible portfolios as $\mathcal{A}_{\mathbb{N}}$ and:

Definition 4.2.1

1. $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$ is *cáglád* and $\mathbb{N}$-adapted

2. for all $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$.

$$H[\int_0^\xi |\vartheta(\tau) - \lambda(\tau)||\beta_1(\tau)\gamma_1(\tau)| + \omega^2(\tau)(\beta_1(\tau)\gamma_1(\tau))^2]d\tau < \infty$$

3. $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$, then $(\beta_1\gamma_1)\,\omega$ is forward integrable and *cáglád* with respect to the Brownian motion D

4. for all $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$, we have $0 < H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)] < \infty$. where $\eta'(x) = \frac{d}{dx}\eta(x)$.

5. for all $\beta_1\gamma_1, \varpi \in \mathcal{A}_{\mathbb{N}}$, there exists $\upsilon > 0$, with ϖ bounded , then $\beta_1\gamma_1 + j\varpi \in \mathcal{A}_{\mathbb{N}}$ for all $j \in (-\upsilon, \upsilon)$, the equation

$$\eta'(X_{\beta_1\gamma_1+j\varpi}(\xi))X_{\beta_1\gamma_1+j\varpi}(\xi)|N_{\beta_1\gamma_1+j\varpi}(\xi)|_{j\in(-\upsilon,\upsilon)}$$

is uniformly integrable, where

$$N_{\beta_1\gamma_1}(\tau) := \int_0^\tau [\vartheta(s) - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]ds + \int_0^t \omega(s)dD(s),\ \tau \in [0, \xi]$$

6. A buy-hold sell strategy ϖ, that is

$$\varpi(t) = \alpha I(\tau, \tau + f](t), \ \ t \in [0, \xi]$$

with $0 \leq \tau < \tau + f \leq \xi$ and α $\mathcal{N}_t$-measurable, belonging to $\mathcal{A}_{\mathbb{N}}$. Then the portfolio $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$ is optimal if

$$H[\eta(X_{\beta_1\gamma_1+j\varpi}(\xi))] = H[\eta(X_{\nu_1}(\xi))]$$

for all $\varpi \in \mathcal{A}_{\mathbb{N}}$ bounded and $j \in (-\upsilon, \upsilon)$ with $\upsilon > 0$ given in (5).

Definition 4.2.2

Assume ϖ is a forward integrable stochastic process and N a random variable. Then the product $N\varpi$ is forward integrable stochastic process and

$$\int_0^{\xi} N\varpi(t)d^{-}D(t) = N\int_0^{\xi} \varpi(t)d^{-}D(t,) \tag{4.3}$$

where $\varpi = X(t)\nu(t)^*$ such that $\nu_1^* = \beta_1\gamma_1$. Firstly, supposing $\beta_1\gamma_1$ is optimal, then for all $\varpi \in \mathcal{A}_{\mathbb{N}}$ bounded implies

$$0 = \frac{d}{dj}H[\eta(X_{\beta_1\gamma_1} + j\varpi(\xi))]|_{j=0}$$

$$\begin{aligned} 0 = H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)\int_0^{\xi} \varpi(s)[\vartheta - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]ds \\ + \int_0^{\xi} \varpi(s)\omega(s)d^{-}D(s)] \end{aligned} \tag{4.4}$$

Now fix $\tau, f : 0 \leq \tau < \tau+f \leq \xi$ and choose $\varpi(s) = \alpha I(\tau, \tau+f](\tau), \ \tau \in [0, \xi]$, where α is an arbitrary bounded and $\mathcal{N}_\tau$-measurable random variable. (4.4) becomes

$$\begin{aligned} 0 = H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)\int_\tau^{\tau+f} [\vartheta - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]ds \\ + \int_\tau^{\tau+f} \omega(s)d^{-}D(s)\alpha] \end{aligned} \tag{4.5}$$

since this holds for all α, we conclude that

$$H[F_{\beta_1\gamma_1}(\xi)(N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau))|_{\mathcal{N}_\tau}] = 0$$

where

$$F_{\beta_1\gamma_1}(\xi) = \frac{\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)}{H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)]}$$

and

$$N_{\beta_1\gamma_1}(\tau) := \exp \int_0^\tau [\vartheta - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]ds] \\ + \int_0^\tau [\omega(s)d^- Ds],\ \tau \in [0, \xi] \tag{4.6}$$

that is,

$$H[F_{\beta_1\gamma_1}(\xi) \int_\tau^{\tau+f} \vartheta(s) - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)ds + \int_\tau^{\tau+f} \omega(s)d^- D(s)|_{\mathcal{N}_\tau}] = 0 \tag{4.7}$$

by the application of Bayes Theorem,

$$H_{Q_{\beta_1\gamma_1}}[N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau)|_{\mathcal{N}_\tau}] \\ 0 = H[F_{\beta_1\gamma_1}(\xi)|_{\mathcal{N}_\tau}]^- H[F_{\beta_1\gamma_1}(\xi)(N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau))|_{\mathcal{N}_\tau}] \tag{4.8}$$

since $N_{\beta_1\gamma_1}(\tau)$ is $\mathcal{N}_\tau$-adapted, this gives

$$H_{Q_{\beta_1\gamma_1}}[N_{\beta_1\gamma_1}(\tau+f)|_{\mathcal{N}_\tau}] = N_{\beta_1\gamma_1}(\tau)$$

Hence $N_{\beta_1\gamma_1}(\tau)$ is an $(\mathcal{N}_\tau, Q_{\beta_1\gamma_1})$-martingale.
Let the probability measure $Q_{\beta_1\gamma_1}$ on $\mathcal{N}_\tau$ be

$$dQ_{\beta_1\gamma_1} = F_{\beta_1\gamma_1}(\xi)dm$$

and set $H_{Q_{\beta_1\gamma_1}}(\tau)$ to
represent $Q_{\beta_1\gamma_1}$ expectation, then

$$H[F_{\beta_1\gamma_1}(\xi)(N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau))|_{\mathcal{N}_\tau}] = 0$$

written as

$$H_{Q_{\beta_1\gamma_1}}[N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau)|_{\mathcal{N}_\tau}] = 0.$$

Therefore, what we have just proved is invariably stated as $N_{\beta_1\gamma_1}(\tau),\ \tau \in [0, \xi]$, is a $(\mathbb{N}, Q_{\beta_1\gamma_1})$-martingale, that is a martingale with respect to the filtration $\mathbb{N}$ under the measure $Q_{\beta_1\gamma_1}$ Conversely, assuming $N_{\beta_1\gamma_1}$ is a
$\mathbb{N}, Q_{\beta_1\gamma_1}$-martingale, then

$$H_{Q_{\beta_1\gamma_1}}[N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau)_{\mathcal{N}_\tau}] = 0$$

for all τ, f therefore $0 \leq \tau < \tau + f \leq \xi$. Equivalently,

$$H_{Q_{\beta_1\gamma_1}}[N_{\beta_1\gamma_1}(\tau+f) - N_{\beta_1\gamma_1}(\tau)\alpha] = 0.$$

for all α bounded $\mathcal{N}_\tau$-measurable. Thus,

$$0 = H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)\int_\tau^{\tau+f}[\vartheta - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]dt \\ + \int_\tau^{\tau+f}\omega(s)d^-D(s)\alpha] \tag{4.9}$$

holds. Taking linear combination, it is valid for all step processes $\varpi \in \mathcal{A}_\mathbb{N}$ of cáglád. Referencing assumption (1) and (5) of definition (4.2.1)

$$0 = H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)\int_0^\xi \varpi(s)[\vartheta - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]ds \\ + \int_0^\xi \varpi(s)\omega(s)d^-D(s)] \tag{4.10}$$

holds with boundedness of all $\varpi \in \mathcal{A}_\mathbb{N}$. Since

$$j \to H[\eta(X_{\beta_1\gamma_1+j\varpi}(\xi))],\ j \in (-\nu,\nu)$$

maximum is obtain at $j = 0$. Thus,

$$0 = \frac{d}{dj}H[\eta(X_{\beta_1\gamma_1+j\varpi}(\xi))]|_{j=0}$$

Definition 4.2.3

If the stochastic process $\beta_1\gamma_1 \in \mathcal{A}_\mathbb{N}$ is optimal for the problem

$$\sup_{\beta_1\gamma_1\in\mathcal{A}_\mathbb{N}} H[\eta(L_{\beta_1\gamma_1}(\xi))] = H[\eta(L^*_{\beta_1\gamma_1}(\xi))]$$

then the stochastic process

$$N_{\beta_1\gamma_1}(\tau) := \exp\int_0^\tau [\vartheta - \lambda(s) - \omega^2(s)\beta_1\gamma_1(s)]ds \\ + \int_0^\tau \omega(s)d^-D(s)] \tag{4.11}$$

is an $(\mathbb{N}, Q_{\beta_1\gamma_1})$-martingale . Conversely, if the function
$g(j) := H[\eta(X_{\beta_1\gamma_1+j\varpi}(\xi))], j \in (-\nu,\nu)$, is concave for each $\varpi \in \mathcal{A}$ and $N_{\beta_1\gamma_1}(\tau)$, $\tau \in [0,\xi]$ is an $(\mathbb{N}, Q_{\beta_1\gamma_1})$-martingale, then $\beta_1\gamma_1 \in \mathcal{A}$ is optimal for the problem

$$\sup_{\beta_1\gamma_1\in\mathcal{A}_\mathbb{N}} H[\eta(L_{\beta_1\gamma_1}(\xi))] = H[\eta(L^*_{\beta_1\gamma_1}(\xi))]$$

$$U(\tau) : \nu_1 \to \lambda(\tau) + (\vartheta(\tau,\nu) - \lambda(\tau))\nu - \frac{1}{2}\omega^2(\tau)\nu^2$$

$$\lambda(\tau) + \vartheta(\tau,\nu)\nu - \lambda(\tau)\nu - \frac{1}{2}\omega^2(\tau)\nu^2$$

$$\vartheta(\tau,\nu)\times 1+\nu\times\vartheta'(\tau,\nu)-\lambda(\tau)\times 1+\nu\times 0-\frac{1}{2}\omega^2(\tau)\times 2\nu+0\times\nu^2$$

$$\vartheta(\tau,\nu)\times 1+\vartheta'(\tau,\nu)\nu-\lambda(\tau)-\omega^2(\tau)\nu$$

upon differentiation, we have

$$\vartheta'(\tau,\nu)+\vartheta''(\tau,\nu)\nu+\vartheta'(\tau,\nu)\times 1-\omega(\tau)$$

$$\vartheta''(\tau,\nu)\nu+2\vartheta'(\tau,\nu)-\omega^2(\tau)\leq 0.$$

we equally have the following result.

Theorem 4.2.1

1. $\beta_1\gamma_1\in\mathcal{A}_{\mathbb{N}}$ is optimal for the equation below

$$\sup_{\beta_1\gamma_1\in\mathcal{A}_{\mathbb{N}}} H[\eta(L_{\beta_1\gamma_1}(\xi))]=H[\eta(L^*_{\beta_1\gamma_1}(\xi))]$$

   only if

$$N_{\beta_1\gamma_1}(s):=N_{\beta_1\gamma_1}(s)-\int_0^s\frac{d[N_{\beta_1\gamma_1},\varsigma_{\beta_1\gamma_1}](\tau)}{\varsigma_{\beta_1\gamma_1}(\tau)},\ s\in[0,\xi]$$

   is an $(\mathbb{N},m)$-martingale. In this regard

$$\varsigma(s):=H_{Q_{\beta_1\gamma_1}}\left[\frac{dm}{dQ_{\beta_1\gamma_1}}|_{\mathcal{N}_s}\right]=(H[F_{\beta_1\gamma_1}(\xi)|\mathcal{N}_s])^{-1},\ s\in[0,\xi]$$

   with

$$F_{\beta_1\gamma_1}(\xi)=\frac{\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)}{H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)]}$$

   and (4.6), (4.9),(4.10) respectively.

2. However, assuming an optimal portfolio $\beta_1\gamma_1\in\mathcal{A}$ exists, then

$$Z(\tau)=\int_0^\tau\omega(s)d^-D(s)$$

   is a semimartingale of $(\mathbb{N},m)$

3. Supposing an optimal $\beta_1\gamma_1\in\mathcal{A}$ exists such that $\omega\neq 0$ for

$$a.a(s,w)\in[0,\xi]\times\Omega$$

   then $D(t)$ is an $(\mathbb{N},m)$-semimartingale

Proof

1. If $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$ is optimal, then by Definition 4.2.3,
   $N_{\beta_1\gamma_1}$ is a $(\mathbb{N}, Q_{\beta_1\gamma_1})$-martingale with
   $$F_{\beta_1\gamma_1}(\xi) = \frac{\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)}{H[\eta'(X_{\beta_1\gamma_1}(\xi))X_{\beta_1\gamma_1}(\xi)]}$$
   and (4.6), (4.9),(4.10) respectively. Thus,
   $$dm(w) = N_{\beta_1\gamma 1}(\tau)dQ_{\beta_1\gamma 1}(w)$$
   on $\mathcal{N}_\xi$
   $$N_{\beta_1\gamma_1}(\tau) = F^{-1}_{\beta_1\gamma_1}$$
   And by Girsanov,we have that
   $$N_{\beta_1\gamma_1}(\tau) := N_{\beta_1\gamma_1}(\tau) - \int_0^\tau \frac{d[N_{\beta_1\gamma_1}, \varsigma](s)}{\varsigma(s)},\ \tau \in [0, \xi]$$
   is an $(\mathbb{N}, m)$-martingale with $\varsigma(\tau)$ an $(\mathbb{N}, Q_{\beta_1\gamma_1})$-martingale stated as
   $$\varsigma(\tau) := H_{Q_{\beta_1\gamma_1}}[\frac{dm}{dQ_{\beta_1\gamma_1}}]|_{\mathcal{N}_\tau} = H[(F_{\beta_1\gamma_1}(\xi))^{-1}\frac{F_{\beta_1\gamma_1}(\xi)}{H[F_{\beta_1\gamma_1}(\xi)|_{\mathcal{N}_\tau}|_{\mathcal{N}_\tau}]}]$$
   $$= (H[F_{\beta_1\gamma_1}(\xi)|\tilde{\mathcal{N}_\tau}])^{-1}\tau \in [0, \xi]$$
   conversely, if $N_{\beta_1\gamma_1}$ is an $(\mathbb{N}, Q_{\beta_1\gamma_1})$-martingale,
   then $N_{\beta_1\gamma_1}$ is an $(\mathbb{N}, m_{\beta_1\gamma_1})$-martingale,thus $\beta_1\gamma_1$ is optimal by Definition 4.2.3

2. is derived from (1)

3. By (2) recall that
   $$Z(\tau) = \int_0^\tau \omega(s)d^- D(s)$$
   is a semimartingale of $(\mathbb{N}, m)$. Then assuming $\omega \neq 0$ for
   $$a.a(t, w) \in [0, \xi] \times \Omega$$
   holds, we obtain
   $$\int_0^\tau \omega^-(s)dZ(s) = \int_0^\tau \omega^-(s)\omega(s)d^- D(s) = D(s)$$
   is a martingale of $(\mathbb{N}, m)$ also.

Theorem 4.2.1 explicitly represents the connection between $\beta_1\gamma_1$ optimal as well as the decomposition of the semimartingale D with respect to $\mathbb{N}$. We prove this in the context of portfolio diversification.

Theorem 4.2.2

1. Given that $\beta_1\gamma_1$ is optimal for

$$\sup_{\beta_1\gamma_1\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{\beta_1\gamma_1}(\xi))] = H[\eta(X^*_{\beta_1\gamma_1}(\xi))]$$

   Then D is a semimartingale with respect to $\mathbb{N}$ with a decomposition

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\beta_1\gamma_1(\tau) - \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{\beta_1\gamma_1},\varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

   where $\hat{D}$ is a $(\mathbb{N}, m)$-Brownian motion

2. In reverse, assume D is a semimartingale with respect to $(\mathbb{N}, m)$ with a decomposition $dD(\tau) = d\hat{D}(\tau) + dA(\tau)$ and A is a $\mathbb{N}$-adapted finite variation process. Assuming $\alpha(\tau)d\tau = dA(\tau) =$ for some $\mathbb{N}$-adapted process α. That is, $dA(\tau)$ is absolutely continuous with respect $d\tau$, where $\hat{D}$ is a $(\mathbb{N}, m)$-Brownian motion then the solution $\beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$ of

$$\omega(\tau)\beta_1\gamma_1 + \frac{1}{\omega(\tau)\varsigma(\tau)}\frac{d\,[N_{\beta_1\gamma_1},\varsigma]\,(\tau)}{d\tau}$$

$$= \alpha(\tau) + \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}.$$

Then, $\beta_1\gamma_1$ is optimal for

$$\sup_{\beta_1\gamma_1\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{\beta_1\gamma_1}(\xi))] = H[\eta(X^*_{\beta_1\gamma_1}(\xi))]$$

since the quadratic variation $N_{\beta_1\gamma_1}$ is continuous absolutely.

$$[N_{\beta_1\gamma_1}, N_{\beta_1\gamma_1}](\tau) = \int_0^\tau \omega^2(s)ds$$

from

$$\hat{N}_{\beta_1\gamma_1}(\tau) := N_{\beta_1\gamma_1}(\tau) - \int_0^\tau \frac{d[N_{\beta_1\gamma_1},\varsigma](s)}{\varsigma(s)},\ \tau\in[0,\xi]$$

this implies $d[N_{\beta_1\gamma_1},\varsigma](\tau)$ is continuous absolutely with respect to $d\tau$. Thus,

$$\frac{d[N_{\beta_1\gamma_1},\varsigma](\tau)}{\omega(\tau)\varsigma(\tau)} = \frac{1}{\omega(\tau)\varsigma(\tau)}\frac{d[N_{\beta_1\gamma_1},\varsigma](\tau)d\tau}{d\tau}$$

Proof

Assuming $\beta_1\gamma_1$ is optimal, by Theorem 4.2.2, the equation below

$$\begin{aligned}\omega^{-1}(\tau)d\hat{N}_{\beta_1\gamma_1}(\tau) = dD(\tau) + \omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\beta_1\gamma_1(\tau))d\tau \\ -\frac{d[N_{\beta_1\gamma_1},\varsigma](\tau)}{\varsigma(\tau)}]\end{aligned} \tag{4.12}$$

$\tau \in [0,\xi]$ is an $\mathbb{N},m$-martingale. Since the quadratic variation of the expression below $\omega^{-1}(\tau)d\hat{N}_{\beta_1\gamma_1}(\tau)$ $\tau \in [0,\xi]$, is τ, $\tau \in [0,\xi]$, it follows that

$$d\hat{D}(\tau) := \omega^{-1}(\tau)d\hat{N}_{\beta_1\gamma_1}(\tau), \quad \tau \in [0,\xi]$$

is a $\mathbb{N},m$-Brownian motion and we have

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\beta_1(\tau)\gamma_1(\tau) - \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{\beta_1\gamma_1}\varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

1. Supposed the decomposition of D is $\mathbb{N},m$-semimartingale ,

   $$dD(\tau) = d\hat{D}(\tau) + dA(\tau),$$

   with reference to (2), we set ν to be

   $$\nu = \alpha(\tau) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}.$$

   Then,

   $$\begin{aligned}\omega^{-1}(\tau)d\hat{N}_{\beta_1\gamma_1}(\tau) = dD(\tau) + \omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\beta_1(\tau)\gamma_1(\tau))d\tau \\ -\frac{d[N_{\beta_1\gamma_1},\varsigma](\tau)}{\varsigma(\tau)}]\end{aligned} \tag{4.13}$$

   $\tau \in [0,\xi]$

   $$dD(\tau) - dA(\tau) = d\hat{D}(\tau)$$

   therefore

   $$d\hat{D}(\tau) = \omega^{-1}(\tau)d\hat{N}_{\beta_1\gamma_1}(\tau), \ \tau \in [0,\xi],$$

   is an $(\mathbb{N},m)$-martingale. Thus, $\beta_1\gamma_1$ is optimal.

For example, Supposing

$$\eta(s) = \log(x), \quad x > 0$$

1. We define a^* as

$$\nu^*(s) = \frac{\vartheta(s) - \lambda(s)}{\omega^2(s)} + \frac{a^*(s)}{\omega(s)}$$

and set

$$D(s) = \frac{\vartheta(s) - \lambda(s)}{\omega(s)}.$$

Then,

$$\hat{D}(t) := D(t) - \int_0^t a^*(s)ds$$

is $(\mathbb{N}, m)$ Brownian motion

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}(D(s) + a^*(s))^2\}ds] \tag{4.14}$$

2. Assuming $D(s)$ is $\mathcal{N}_s$-measurable $\xi \geq 0$. Then,

$$H[\int_0^\xi D(s)a^*(s)ds] = 0$$

and the similar value is

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \log x_0 + H[\int_0^\xi \{\lambda(s) \\ &+ \frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds] \end{aligned} \tag{4.15}$$

Proof

In as much as ν^* is admissible, therefore, an equivalent optimal value function

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \log x_0 + H[\int_0^\xi \{\lambda(s) \\ &+ \frac{1}{2}(B(s) + a^*(s))^2\}ds] \end{aligned} \tag{4.16}$$

is finite. We prove that

$$H\left[\int_0^\xi D(s)a^*(s)ds\right] = 0.$$

Suppose $D(s)$ is $\mathcal{N}$-adapted, then by $A(t) := D(t) - \hat{D}(t)$, we have

$$\begin{aligned} H\left[\int_0^\xi D(s)a^*(s)ds\right] &= H\left[\int_0^\xi D(s)(dD(s) - d\hat{D}(s))\right] \\ &= H\left[\int_0^\xi D(s)dD(s)\right] - H\left[\int_0^\xi D(s)d\hat{D}(s)\right] = 0. \end{aligned} \tag{4.17}$$

From

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds].$$

Suppose

$$(a^*(s)) = \beta_1\gamma_1.$$

This equally means that our mode of investment can be diversified

$$\frac{1}{2}\left(\beta_1\gamma_1 + D(s)\right)^2 = (\beta_1\gamma_1 + D(s))(\beta_1\gamma_1 + D(s))$$

$$\beta_1^2\gamma_1^2 + \beta_1\gamma_1 D(s) + \beta_1\gamma_1 D(s) + D(s)^2$$

$$\frac{1}{2}[\beta_1^2\gamma_1^2 + 2\beta_1\gamma_1 D(s) + D(s)^2]$$

$$\frac{1}{2}[D(s)^2 + \beta_1^2\gamma_1^2 + \beta_1\gamma_1 D(s)].$$

Hence, from

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds]$$

we have,

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[(D(s)^2 + (2\beta_1^2\gamma_1^2) + (2\beta_1\gamma_1 D(s))]\}ds]$$

since ν is admissible, then the corresponding optimal value function in (4.24) is finite. Consequently, if $D(s)$ is $\mathcal{N}_s$-measurable, $0 \le s \le \xi$. Then

$$H[\int_0^\xi \beta_1\gamma_1 D(s)ds] = 0$$

By $A(t) := D(t) - \hat{D}(t)$ being an $\mathcal{N}_t$-Brownian motion, therefore,

$$H[\int_0^\xi D(s)\beta_1\gamma_1 ds] = H[\int_0^\xi D(s)\left(dD(s) - d\hat{D}(s)\right)]$$

$$= H[\int_0^\xi D(s)dD(s)] - H[\int_0^\xi D(s)d\hat{D}(s)] = 0$$

1. Given $\eta(x) = \frac{1}{h}x^h,\ \ x > 0$, where $h \in (0,1)$, we have

$$\eta'(X_{\beta_1\gamma_1+h\varpi}(\xi))X_{\beta_1\gamma_1+h\varpi}(\xi)|M(h)| = X_{\beta_1\gamma_1^h+h\varpi}(\xi)|M(h)$$

   and condition (4) in our earlier Definition is satisfied if

$$\sup_{h\in(-\delta,\delta)} H[(X^h_{\beta_1\gamma_1+h\varpi}(\xi)|M(h)|)^{\hat{p}}] < \infty$$

for $\hat{p} > 1$, then set

$$X_{\beta_1\gamma_1+h\varpi}(\xi) = X_{\beta_1\gamma_1}(\xi)N(h).$$

From the Holders inequality,

$$H[(X^h_{\beta_1\gamma_1+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{\beta_1\gamma_1}(\xi))^{h\hat{p}\tilde{a}_1\tilde{b}_1}])^{\frac{1}{\tilde{a}_1\tilde{b}_1}}$$

$$(H[(N(h))^{h\hat{p}\tilde{a}_1\tilde{b}_2}])^{\frac{1}{\tilde{a}_1\tilde{b}_2}}(H[(|M(h)|)^{\hat{p}\tilde{a}_2}])^{\frac{1}{\tilde{a}_2}}$$

where $\tilde{a}_1, \tilde{a}_2 : \frac{1}{\tilde{a}_1} + \frac{1}{\tilde{a}_2} = 1$ and $\tilde{b}_1, \tilde{b}_2 : \frac{1}{\tilde{b}_1} + \frac{1}{\tilde{b}_2} = 1$. Choosing $\tilde{a}_1 = \frac{2}{2-\hat{p}}, \tilde{a}_2 = \frac{2}{\hat{p}}$ and also $\tilde{b}_1 = \frac{2-\hat{p}}{h\hat{p}},\ \tilde{b}_2 = \frac{2-\hat{p}}{2-\hat{p}-h\hat{p}}$ for some $\hat{p} \in (1, \frac{2}{h+1})$. Hence,

$$H[(X^h_{\beta_1\gamma_1+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{\beta_1\gamma_1}(\xi))^2])^{\frac{h\hat{p}}{2}}$$

$$(H[(N(h))^{\frac{2h\hat{p}}{2-\hat{p}-h\hat{p}}}])^{\frac{2-\hat{p}-h\hat{p}}{2}}(H[(|M(h)|^2)])^{\frac{\hat{p}}{2}}.$$

supposing $N_{\beta_1\gamma_1}(\xi)$ in

$$\begin{aligned} N_{\beta_1\gamma_1}(t) = x\exp\{\int_0^\tau [\lambda(\tau) + (\vartheta(\tau) - \lambda(\tau))\beta_1(\tau)\gamma_1(\tau) \\ -\frac{1}{2}\omega^2(\tau)(\beta_1(\tau)\gamma_1(\tau))^2]d\tau + \int_0^\tau \omega(\tau)\beta_1(\tau)\gamma_1(\tau)d^-D(\tau) \end{aligned} \tag{4.18}$$

satisfies

$$H[(N_{\beta_1\gamma_1}(\xi))^2] < \infty.$$

Then, item (4) and item (5) of definition (4.2.1) is valid if

$$\sup_{h\in-(\delta,\delta)} H[(N(h))\frac{2h\hat{p}}{2-\hat{p}-x\hat{p}}] < \infty$$

however,

$$\sup_{h\in-(\delta,\delta)} H[(N(h))\frac{2h\hat{p}}{2-\hat{p}-x\hat{p}}] < \infty$$

holds if for example

$$H\exp\{k\int_0^s [|\vartheta(\tau) - \lambda(\tau)| + |\beta_1\gamma_1(\tau)|d\tau\}] < \infty \ \ \forall\ k > 0$$

however,

$$H[(N_{\beta_1\gamma_1}(\xi))^2] < \infty$$

is equally verify for $k > 0$

$$\begin{aligned} H\exp\{k(\int_0^\xi [|\vartheta(\tau) - \lambda(\tau)| + |\beta_1\gamma_1(\tau)|d\tau])\} \\ +|\int_0^\xi \beta_1\gamma_1(\tau)\omega(\tau)dD(\tau)| < \infty \end{aligned} \tag{4.19}$$

Example 4.2.1

$$U(x) = \frac{1}{c}x^c; \quad x \in [0, \infty),$$

where $c \in (-\infty, 1) - \{0\}$ is a constant. Assuming $\omega \neq 0$ for $a.a\ (t, w)$.

Let $\mathcal{F}_t \subset \mathcal{G}_t$ be a filtration. Assume $\nu^* \in \mathcal{A}$ is an optimal portfolio for the problem

$$\phi_{\mathcal{G}} := \sup_{\nu \in \mathcal{A}} E[\frac{1}{c}(X_{(\nu)}(\xi))^c].$$

Then, there exists an $\mathcal{G}$-adapted process $\alpha(s)$ such that $\hat{D}(t) := D(t) - A(t)$ holds and hence, the optimal sensitive investor portfolio is.

Let the (modify) market price of risk, $\theta(t)$, by

$$\theta(t) = \frac{\vartheta(t) - \lambda(t)}{\omega(t)} + \alpha(t).$$

Assume that

$$E[\exp(\frac{1}{2}\int_0^\xi \theta^2(t)dt)] < \infty$$

and define

$$H_0(t) = \exp(-\int \theta(s)d\hat{D}(s) - \frac{1}{2}\int_0^t (\theta^2(s) + r(s))ds)$$

$$X^*(t) = H_0^{-1}(t)E[H_0^{\frac{c}{c-1}}(\xi)]^{-1}E[H_0^{\frac{c}{c-1}}(\xi)/\mathcal{G}_t]$$

Let $\psi(t)$ be the unique $\mathcal{G}_t$-adapted process such that

$$\int_0^\xi \psi^2(t)dt < \infty$$

a.s. and

$$E[H_0^{\frac{c}{c-1}}(\xi)]^{-1}H_0^{\frac{c}{c-1}}(\xi) = 1 + \int_0^\xi \psi(t)d\hat{D}(t).$$

Then,

$$\nu^*(t) = \omega^{-1}(t)\left[\frac{\psi(t)}{H_0(t)} + X^*(t)\theta(t)\right]$$

is the sensitive investors optimal portfolio and $X^*(t) = X^*_\nu(t)$ is the similar investors optimal value process. However, if the utility function is of the form,

$$U(x) = \frac{x^{\gamma *}}{\gamma}$$

where $\gamma *$ is $\vartheta - \lambda$ and γ is $1 - c$ then, the Optimal portfolio is

$$\nu(t) = \frac{\vartheta - \lambda}{(1-c)\omega^2}$$

$$\frac{0.09 - 0.04}{(1 - 0.8) \times (0.04)}$$

at all times t where $c < 1$

$$\frac{0.09 - 0.04}{0.2 \times 0.04} = \frac{0.05}{0.008} = 6.25$$

i.e. 25% of his money is invested in S and 75% in the risk-free asset, we have that when $\vartheta = 9, \omega = 20\%,\ \lambda = 4\% c = 0.8,$

4.3 Multiple assets diversification with restrictions for investors

The present day world have subjected both the low and high income earners to multiple streams of investment for a better multiple streams of income in return. As a result, a sensitive investor splits his investment network on different assets to avoid less return as expected from assets. we defined

$$H : \mathbb{R}^n \to \langle \beta, \gamma \rangle, \subseteq \mathbb{R}$$

where

$$\langle \hat{\beta, \hat{\gamma}} \rangle = \langle (\beta_1 \beta_2), (\gamma_1 \gamma_2), \cdots \rangle = \sum_{i=1}^{n} \beta_i \gamma_i$$

where $\sum_{i=1}^{n} \nu_i \beta_i \gamma_i$ and ν_i is a confine placed before investors on the size of assets to be held due to costs of transactions. $\nu_i = 1$ if assets i is chosen in the portfolio, and 0 otherwise Mark Schroder, and Costis Skiadas, (1999).

Definition 4.3.1

$\mathcal{A}_{\mathbb{N}}$ is admissible portfolios expressed as

1. all $\sum_{i=1}^{n} \beta_i \gamma_i \in \mathcal{A}_{\mathbb{N}}$ is *cáglád* and $\mathbb{N}$-adapted i.e. $\mathbb{N}$-measurable

2. $\sum_{i=1}^{n} \beta_i \gamma_i \in \mathcal{A}_{\mathbb{N}}$.

$$H[\int_0^{\xi} |\vartheta(\tau) - \lambda(\tau)||\,(\beta_i(\tau)\gamma_i(\tau)|+)\,(\sum_{i=1}^{n} \upsilon \beta_i(\tau)\gamma_i(\tau))^2]d\tau < \infty$$

3. $\beta_i \gamma_i \in \mathcal{A}_{\mathbb{N}}$,the product $\sum_{i=1}^{n} (\beta_i \gamma_i)\, \omega$ is forward integrable and *cáglád*

4. $\sum_{i=1}^{n} \beta_i \gamma_i \in \mathcal{A}_{\mathbb{N}}$, then $0 < H[\eta' L_{\sum_{i=1}^{n} (\beta_i \gamma_i}(\xi)) L_{\sum_{i=1}^{n} \beta_i \gamma_i}(\xi)] < \infty$ where $\eta'(l) = \frac{d}{dl}\eta(l)$.

5. for all $\sum_{i=1}^{n}\beta_i\gamma_i, \varpi \in \mathcal{A}_{\mathbb{N}}$, there exist $\delta > 0$, with a bounded ϖ, such that $\sum_{i=1}^{n}\beta_i\gamma_i + j\varpi \in \mathcal{A}_{\mathbb{N}}$ in all $j \in (-\delta, \delta)$ as such, the entire family

$$\eta'(L_{\sum_{i=1}^{n}\beta_i\gamma_i+j\varpi}(\xi))L_{\sum_{i=1}^{n}\beta_i\gamma_i+j\varpi}(\xi)|N_{\sum_{i=1}^{n}\beta_i\gamma_i+j}(\xi)|_{j\in(-\delta,\delta)}$$

is consistently integrable, where

$$N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) := \int_0^s [\lambda(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^{n}\beta_i\gamma_i(\tau)]d\tau + \int_0^s \omega(s)dD(\tau),$$

6. ϖ portfolio permits buy and keep to a reasonable time of choice before reselling, of

$$\varpi(s) = \alpha I(\tau, \tau + f](t), \quad t \in [0, \xi]$$

with $0 \leq \tau < \tau + f \leq \xi$ and $\alpha\mathcal{N}_\tau$-measurable belonging to $\mathcal{A}_{\mathbb{N}}$. Then $\sum_{i=1}^{n}\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal if

$$H[\eta(X_{\sum_{i=1}^{n}\beta_i\gamma_i+j\varpi}(\xi))] = H[\eta(L_{\nu_1}(\xi))]$$

for a bounded $\varpi \in \mathcal{A}_{\mathbb{N}}$ and $j \in (-\delta, \delta)$ with $\delta > 0$ given in item (5) of definition (4.3.1).

Definition 4.3.2

Assuming ϖ is a forward integrable stochastic process and N a random variable. Then the product $N\varpi$ is a stochastic process and forward integrable then

$$\int_0^\xi N\varpi(\tau)d^-D(\tau) = N\int_0^\xi \varpi(\tau)d^-D(\tau)$$

where $\varpi = X(\tau)\nu(\tau)^*$ in this sense, $\nu_1^* = \sum_{i=1}^{n}\beta_i\gamma_i$. Firstly, $\sum_{i=1}^{n}\beta_i\gamma_i$ is optimal. For a bounded $\varpi \in \mathcal{A}_{\mathbb{N}}$ we have

$$0 = \frac{d}{dj}H[\eta(X_{\sum_{i=1}^{n}\beta_i\gamma_i+j\varpi}(\xi))]|_{j=0}$$

$$\begin{aligned} 0 = H[\eta'(X(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)\int_0^\xi \varpi(\tau)[\vartheta - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^{n}\beta_i\gamma_i(\tau)]d\tau \\ + \int_0^\xi \varpi(\tau)\omega(\tau)d^-D(\tau)] \end{aligned} \tag{4.20}$$

fixing $\tau, f : 0 \leq \tau < \tau + f \leq \xi$ and choosing $\varpi(s) = \alpha I(\tau, \tau + f](s), \ s \in [0, \xi]$, for any possible quantity of fluctuating α of $\mathcal{N}_\tau$ bounded. (4.20) becomes

$$\begin{aligned} 0 = H[\eta'(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)\int_\tau^{\tau+f}[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n}\beta_i\gamma_i(s)]ds \\ + \int_\tau^{\tau+f}\omega(s)d^-D(s)\alpha] \end{aligned} \tag{4.21}$$

it is valid for all α, then

$$H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)(N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f)-N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}]=0$$

where

$$F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)=\frac{\eta'(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)}{H[\eta'(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)]}$$

and

$$\begin{aligned}N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau):=\exp\int_0^\tau[\vartheta-\lambda(s)-\omega^2(s)\sum_{i=1}^{n}\beta_i\gamma_i(s)]ds\\+\int_0^\tau\omega(s)d^-D(s)]\end{aligned}\tag{4.22}$$

that is,

$$\begin{aligned}0=H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)(\int_s^{s+f}\vartheta(\tau)-\lambda(\tau)-\omega^2(\tau)\sum_{i=1}^{n}\beta_i\gamma_i(\tau)\}d\tau\\+\int_s^{s+f}\omega(\tau)d^-D(\tau))|_{\mathcal{N}_\tau}]\end{aligned}\tag{4.23}$$

$$\begin{aligned}H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}[N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f)-N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)|_{\mathcal{N}_\tau}]\\0=H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)|_{\mathcal{N}_\tau}]^{-}H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)(N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f)-N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}]\end{aligned}\tag{4.24}$$

since $N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)$ is $\mathcal{N}_\tau$-adapted, this gives

$$H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}[N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f)|_{\mathcal{N}_\tau}]=N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau).$$

Hence, $N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)$ is an $(\mathcal{N}_\tau, Q_{\sum_{i=1}^{n}\beta_i\gamma_i})$-martingale.
Let the probability measure $Q_{\sum_{i=1}^{n}\beta_i\gamma_i}$ on $\mathcal{N}_\tau$ be

$$dQ_{\sum_{i=1}^{n}\beta_i\gamma_i}=F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)dm$$

and $H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}(\tau)$ an expectation of $Q_{\sum_{i=1}^{n}\beta_i\gamma_i}$ So,

$$H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)(N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f)-N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}]=0$$

written as

$$H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}[N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f)-N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)|_{\mathcal{N}_\tau}]=0$$

$N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau),\ \tau \in [0,\xi]$, is a $(\mathbb{N}, Q_{\sum_{i=1}^{n}\beta_i\gamma_i})$-martingale, that is, a martingale with respect to the filtration $\mathbb{N}$ under the probability$Q_{\sum_{i=1}^{n}\beta_i\gamma_i}$ it can equally be stated as follows that suppose $N_{\sum_{i=1}^{n}\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{\sum_{i=1}^{n}\beta_i\gamma_i})$-martingale. Then,

$$H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}[N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f) - N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)_{\mathcal{N}_\tau}] = 0$$

for all τ, f and then $0 \leq \tau < \tau + f \leq \xi$. similarly,

$$H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}[N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau+f) - N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)\alpha] = 0$$

for all bounded $\mathcal{N}_\tau$-measurable α, therefore

$$\begin{aligned} 0 = H[\eta'(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)\int_{\tau}^{\tau+f}[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n}\beta_i\gamma_i(s)]ds \\ +\int_{\tau}^{\tau+f}\omega(s)d^-D(s)\alpha] \end{aligned} \tag{4.25}$$

holds, taking linear combination

$$\begin{aligned} 0 = H[\eta'(X_{\beta_i\gamma_i}(\xi))X_{\beta_i\gamma_i}(\xi)\int_{0}^{\xi}\varpi(s)[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n}\beta_i\gamma_i(s)]ds \\ +\int_{0}^{\xi}\varpi(s)\omega(s)d^-D(s)] \end{aligned} \tag{4.26}$$

remain valid for all *cáglád* step processes $\varpi \in \mathcal{A}_\mathbb{N}$ from assumption (1) and (5) of definition (4.2.3) we have (4.26)

still holds for all bounded $\varpi \in \mathcal{A}_\mathbb{N}$. Provided the function

$$j \to H[\eta(X_{\sum_{i=1}^{n}\beta_i\gamma_i + j\varpi}(\xi))],\ j \in (-\delta, \delta)$$

maximum is achieve at $j = 0$. Thus,

$$0 = \frac{d}{dj}H[\eta(X_{\sum_{i=1}^{n}\beta_i\gamma_i + j\varpi}(\xi))]|_{j=0}$$

Definition 4.3.3

A stochastic process $\sum_{i=1}^{n}\beta_i\gamma_i \in \mathcal{A}_\mathbb{N}$ is optimal as far as the stochastic processs

$$\begin{aligned} N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) := \exp\int_{0}^{\tau}[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n}\beta_i\gamma_i(s)]ds \\ +\int_{0}^{\tau}\omega(s)d^-D(s)],\ \tau \in [0,\xi] \end{aligned} \tag{4.27}$$

is an $\mathbb{N}, Q_{\sum_{i=1}^{n}\beta_i\gamma_i}$-martingale. By the application of theorem of Girsanov, it is equally stated as follows

Theorem 4.3.1

1. The process $\sum_{i=1}^{n}\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ for

$$\sup_{\sum_{i=1}^{n}\beta_i\gamma_i\in\mathcal{A}_{\mathbb{N}}} H[\eta(L_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))] = H[\eta(L^*_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))]$$

is optimal if

$$N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) := N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) - \int_0^\tau \frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \upsilon_{\sum_{i=1}^{n}\beta_i\gamma_i}](s)}{\upsilon_{\sum_{i=1}^{n}\beta_i\gamma_i}(s)}$$

for $\tau \in [0,\xi]$ is a $(\mathbb{N}, m)$-martingale in the sense that,

$$\upsilon_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) := H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}\left[\frac{dm}{dQ_{\sum_{i=1}^{n}\beta_i\gamma_i}}|_{\mathcal{N}_\tau}\right]$$

$$= (H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)|\mathcal{N}_\tau])^{-}1, \tau \in [0,\xi]$$

2. However, If an optimal portfolio $\sum_{i=1}^{n}\beta_i\gamma_i \in \mathcal{A}$ exists, hence the process

$$\varsigma(s) = \int_0^\tau \sigma(s)d^- D(s)$$

is a $(\mathcal{N}, m)$-semimartingale

3. supposed $\sum_{i=1}^{n}\beta_i\gamma_i \in \mathcal{A}$ of optimal value exists and $\sigma \neq 0$ for

$$a.a(s,w) \in [0,\xi] \times \Omega$$

then $D(\tau)$ is an $(\mathbb{N}, m)$-semimartingale

Proof

1. If $\sum_{i=1}^{n}\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$

   is optimal, by Definition 4.3.3, then $N_{\sum_{i=1}^{n}\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{\sum_{i=1}^{n}\beta_i\gamma_i})$-martingale with

$$F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi) = \frac{\eta'(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)}{H[\eta'(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)]}$$

   and (4.22). By applying theorem of Girsanov,

$$\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) := N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) - \int_0^\tau \frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](s)}{\varsigma(s)}$$

is a $(\mathbb{N}, m)$-martingale

$$\varsigma(\tau) := H_{Q_{\sum_{i=1}^{n}\beta_i\gamma_i}}[\frac{dm}{dQ_{\sum_{i=1}^{n}\beta_i\gamma_i}}]|_{\mathcal{N}_\tau}$$

$$= H[(F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))^{-}1\frac{F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)}{H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)|_{\mathcal{N}_\tau}|_{\mathcal{N}_\tau}]}]$$

$$= (H[F_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)|\mathcal{N}_\tau])^{-1}, \tau \in [0, \xi]$$

conversely, if $N_{\sum_{i=1}^{n}\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{\beta_i\gamma_i})$-martingale then $N_{\sum_{i=1}^{n}\beta_i\gamma_i}$ is a $(\mathbb{N}, m_{\beta_i\gamma_i})$-martingale thus $\sum_{i=1}^{n}\beta_i\gamma_i$ is optimal by Definition 4.3.3

2. is the direct value of (1)

3. By (2) its obvious that

$$Z(s) = \int_0^s \omega(\tau)d^{-}D(\tau)$$

is a $(\mathbb{N}, m)$-semimartingale. supposing $\omega \neq 0$ for

$$a.a(\tau, w) \in [0, \xi] \times \Omega$$

its valid, then

$$\int_0^s \omega^{-}(\tau)dZ(\tau) = \int_0^s \omega^{-}(\tau)\omega(\tau)d^{-}D(\tau) = D(\tau)$$

is $(\mathbb{N}, m)$-semimartingale also.

Theorem 4.3.1 gives a clear connection between $\sum_{i=1}^{n}\beta_i\gamma_i$ optimal portfolio and decomposition of semimartingale D with respect to $\mathbb{N}$. we prove this in the context of portfolio diversification.

Theorem 4.3.2

1. Given that $\sum_{i=1}^{n}\beta_i\gamma_i$ is optimal, then the decomposition of semimartingale D with respect to $\mathbb{N}$ is

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\sum_{i=1}^{n}\beta_i\gamma_i(\tau) - \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

where $\hat{D}$ is Brownian motion of $(\mathbb{N}, m)$-Brownian motion

2. In reverse, assume the semimartingale D with respect to $(\mathbb{N}, m)$ decomposes as $dD(\tau) = d\hat{D}(\tau) + dA(\tau)$, where $\hat{D}$ is $(\mathbb{N}, m)$ Brownian and $\mathbb{N}$-adapted finite variation process A. Assume $dA(\tau) = \alpha(\tau)d(\tau)$ for some $\mathbb{N}$-adapted process α that is, $dA(t)$ is absolutely continuous with respect to $d(t)$ then there is a solution $\sum_{i=1}^{n} \beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ of the form

$$\omega(\tau)\sum_{i=1}^{n}\beta_i\gamma_i + \frac{1}{\omega(\tau)\varsigma(\tau)}\frac{d\left[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma\right](\tau)}{d\tau}$$

$$= \alpha(\tau) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}$$

in this sense $\sum_{i=1}^{n} \beta_i\gamma_i$ proves optimal for

$$\sup_{\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}} H[\eta(X_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))]$$

since quadratic variation of $\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}$ is absolutely continuous that is,

$$[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, N_{\sum_{i=1}^{n}\beta_i\gamma_i}](\tau) = \int_0^\tau \omega^2(s)ds$$

from

$$\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) := N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) - \int_0^\tau \frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](s)}{\varsigma(s)},\ \tau \in [0, \xi]$$

then $d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](\tau)$ is absolutely continuous with respect to $d\tau$.

$$\frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](\tau)}{\omega(\tau)\varsigma(\tau)} = \frac{1}{\omega(\tau)\varsigma(\tau)}\frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](\tau)d\tau}{d\tau}$$

Proof

Assuming $\sum_{i=1}^{n} \beta_i\gamma_i$ is optimal, by Theorem 4.3.1,

$$\begin{aligned}\omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) = dD(\tau)\\ +\omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^{n}\beta_i\gamma_i(\tau))d\tau\\ -\frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}\varsigma](\tau)}{\varsigma(\tau)}]\end{aligned} \tag{4.28}$$

$\tau \in [0, \xi]$ is an $(\mathbb{N}, m)$-martingale and $\omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau)$ for $\tau \in [0, \xi]$, is $\tau, \tau \in [0, \xi]$ a quadratic variation then

$$d\hat{D}(\tau) := \omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau),\ \ \tau \in [0, \xi]$$

is an $(\mathbb{N}, m)$-Brownian motion and

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\sum_{i=1}^{n}\beta_i(\tau)\gamma_i(\tau) - \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

in effect

1. Assume D is a $(\mathbb{N}, m)$-semimartingale with a decomposition

$$dD(\tau) = d\hat{D}(\tau) + dA(\tau)$$

with referencing to (2). Set ν

$$\nu = \alpha(\tau) + \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}.$$

Then,

$$\begin{aligned} \omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau) = dD(\tau) \\ +\omega^{-1}(\tau)[(\vartheta(\tau)-\lambda(\tau)-\omega^2(\tau)\sum_{i=1}^{n}\beta_i(t)\gamma_i(\tau))d\tau \\ -\frac{d[N_{\sum_{i=1}^{n}\beta_i\gamma_i}, \varsigma](\tau)}{\varsigma(\tau)}] \end{aligned} \tag{4.29}$$

$\tau \in [0, \xi]$

$$= dD(\tau) - dA(\tau) = d\hat{D}(\tau)$$

consequently

$$d\hat{D}(\tau) = \omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n}\beta_i\gamma_i}(\tau),\ \tau \in [0, \xi],$$

is a $(\mathbb{N}, m)$-martingale then $\sum_{i=1}^{n}\beta_i\gamma_i$ is optimal from Theorem 4.3.2

For example, Assuming

$$U(s) = \log(x), \quad x > 0$$

1. We define a^* as

$$\nu^*(\tau) = \frac{\vartheta(\tau)-\lambda(\tau)}{\omega^2(\tau)} + \frac{a^*(\tau)}{\omega(\tau)}$$

and let

$$D(\tau) = \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}.$$

Then,

$$\hat{D}(s) := D(s) - \int_0^\tau a^*(t)dt$$

is a $(\mathcal{N}, m)$-Brownian motion and

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(\tau) + \frac{1}{2}(D(\tau) + a^*(\tau))^2\}d\tau] \tag{4.30}$$

2. Assume that $D(\tau)$ is $\mathcal{F}_\tau$-measurable, $\xi \geq 0$. Then,

$$H[\int_0^\xi D(\tau)a^*(\tau)d\tau] = 0$$

and the similar value is

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \exp x_0 + H[\int_0^\xi \{\lambda(s) \\ &+\frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds] \end{aligned} \tag{4.31}$$

Proof

Since ν^* is admissible, the corresponding optimal value function

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \log x_0 + H[\int_0^\xi \{\lambda(s) \\ &+\frac{1}{2}(D(s) + a^*(s))^2\}ds] \end{aligned} \tag{4.32}$$

is finite. We prove

$$H\left[\int_0^\xi D(\tau)a^*(\tau)d\tau\right] = 0$$

by saying that if $D(s)$ is $\mathcal{F}$-adapted, then by $\hat{D}(t) := D(t) - A(t)$

$$\begin{aligned} H\left[\int_0^\xi D(s)a^* ds\right] &= H\left[\int_0^\xi D(s)\left(Dd(s) - d\hat{D}(s)\right)\right] \\ &= H\left[\int_0^\xi D(s)dD(s)\right] - H\left[\int_0^\xi D(s)d\hat{D}(s)\right] = 0. \end{aligned} \tag{4.33}$$

From

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(\tau) + \frac{1}{2}[D(\tau)^2 + (a^*(\tau))^2]\}d\tau]$$

Let

$$(a^*(s)) = \sum_{i=1}^n \beta_i\gamma_i.$$

This equally means that our mode of investment can be diversified

$$\frac{1}{2}\sum_{i=1}^n (\beta_i\gamma_i + D(s))^2 = \sum_{i=1}^n (\beta_1\gamma_1 + D(s)) \sum_{i=1}^n (\beta_i\gamma_i + D(s))$$

$$\sum_{i=1}^n \beta_i^2\gamma_i^2 + \sum_{i=1}^n \beta_i\gamma_i D(s) + \sum_{i=1}^n \beta_i\gamma_i D(s) + D(s)^2.$$

Hence, from

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds]$$

we have

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[(D(s)^2 + \sum_{i=1}^n(\beta_i^2\gamma_i^2) + \sum_{i=1}^n(2\beta_i\gamma_i D(s))]\}ds]$$

is finite. As a result, $D(s)$ is $\mathcal{N}_s$-measurable, $0 \le t \le \xi$. Therefore, $H[\int_0^\xi \sum_{i=1}^n \beta_i\gamma_i D(s)ds] = 0$ By $\hat{D}(\tau) := D(\tau) - A(\tau)$ being $\mathcal{N}_t$-Brownian motion, we have

$$H[\int_0^\xi D(s)\sum_{i=1}^n \beta_i\gamma_i ds] = H[\int_0^\xi D(s)\left(dD(s) - d\hat{D}(s)\right)]$$

$$= H[\int_0^\xi D(s)dD(s)] - H[\int_0^\xi D(s)d\hat{D}(s)] = 0$$

1. Given $U(x) = \frac{1}{y}x^y,\ \ x > 0$ where $y \in (0,1)$ we have

$$\eta'(X_{\sum_{i=1}^n \beta_i\gamma_i + y\varpi}(\xi))X_{\sum_{i=1}^n \beta_i\gamma_i + y\varpi}(\xi)|M(y)| = X^y_{\sum_{i=1}^n \beta_i\gamma_i} + y\varpi(\xi)|M(y)$$

and condition (4) in our earlier Definition is satisfied if

$$\sup_{y\in(-\delta,\delta)} H[(X^y_{\sum_{i=1}^n \beta_i\gamma_i + y\varpi}(\xi)|M(y)|)^p] < \infty$$

for $p > 1$ set

$$X_{\sum_{i=1}^n \beta_i\gamma_i + y\varpi}(\xi) = X_{\sum_{i=1}^n \beta_i\gamma_i}(\xi)N(y)$$

where

$$N_{\sum_{i=1}^n \beta_i\gamma_i}(y) := \exp\int_0^\xi [\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^n \beta_i\gamma_i(\tau)]d\tau + \int_0^t \omega)dD(\tau).$$

From the Holders inequality, we have

$$H[(X^y_{\sum_{i=1}^n \beta_i\gamma_i + y\varpi}(\xi)|M(y)|)^{\hat{p}}] \le (H[(X_{\sum_{i=1}^n \beta_i\gamma_i}(\xi))^{y\hat{p}\ddot{a}_1 b_1}])^{\frac{1}{\ddot{a}_1 b_1}}$$

$$(H[(N(y))^{y\hat{p}\ddot{a}_1 b_2}])^{\frac{1}{\ddot{a}_1 b_2}}(H[(|M(y)|)^{\hat{p}\ddot{a}_2}])^{\frac{1}{\ddot{a}_2}}$$

where $\ddot{a}_1, \ddot{a}_2 : \frac{1}{\ddot{a}_1} + \frac{1}{\ddot{a}_2} = 1$ and $b_1, b_2 : \frac{1}{b_1} + \frac{1}{b_2} = 1$. Then, we set $\ddot{a}_1 = \frac{2}{2-p}, \ddot{a}_2 = \frac{2}{\hat{p}}$ and also $b_1 = \frac{2-\hat{p}}{y\hat{p}},\ b_2 = \frac{2-\hat{p}}{2-\hat{p}-y\hat{p}}$ for some $\hat{p} \in (1, \frac{2}{y+1})$. Hence,

$$H[(X^y_{\sum_{i=1}^n \beta_i\gamma_i} + y\varpi(\xi)|M(y)|)^{\hat{p}}] \le (H[(X_{\sum_{i=1}^n \beta_i\gamma_i}(xi))^2])^{\frac{yp}{2}}$$

$$(H[(N(y))^{\frac{2y\hat{p}}{2-\hat{p}-y\hat{p}}}])^{\frac{2-\hat{p}-y\hat{p}}{2}}(H[(|M(y)|^2)])^{\frac{\hat{p}}{2}}$$

if $N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi)$ in

$$\begin{aligned} N(s) = x\exp\{\int_0^s \lambda(\tau) + (\vartheta(\tau) - \lambda(\tau))\sum_{i=1}^{n}\beta_i(\tau)\gamma_i(\tau) \\ -\frac{1}{2}\omega^2(\tau)(\sum_{i=1}^{n}\beta_i(\tau)\gamma_i(\tau))^2]d\tau + \int_0^s \omega(\tau)\sum_{i=1}^{n}\beta_i(\tau)\gamma_i(\tau)d^- D(\tau) \end{aligned} \tag{4.34}$$

satisfies

$$H[(N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))^2] < \infty$$

Then the condition (4) and (5) of our earlier definitions holds if

$$\sup_{y\in -\delta,\delta} H[(N(y))\frac{2yp}{2-p-xp}] < \infty$$

however

$$\sup_{y\in -\delta,\delta} H[(N(y))\frac{2yp}{2-p-xp}] < \infty$$

holds if for example

$$H\exp\{k\int_0^{\tau}[|\vartheta(\tau) - \lambda)| + |\sum_{i=1}^{n}\beta_i\gamma_i(\tau)|d\tau\}] < \infty \;\forall\; k > 0$$

however,

$$H[(N_{\sum_{i=1}^{n}\beta_i\gamma_i}(\xi))^2] < \infty$$

is equally verify for all $k > 0$

$$\begin{aligned} H\exp\{k(\int_0^{\xi}[|\vartheta(\tau) - \lambda(\tau)| + |\sum_{i=1}^{n}\beta_i\gamma_i(\tau)|d\tau)\} \\ +|\int_0^{\xi}\sum_{i=1}^{n}\beta_i\gamma_i(\tau)\omega(\tau)dD(\tau)| < \infty \end{aligned} \tag{4.35}$$

4.4 Characteristics of optimal portfolio of a sensitive investor with insurance cover

Effective portfolio management entails effective investing, monitoring the market trends against spontaneous shift in the economy and changes to the political scenario as well as issues that may influence some organisations. The risk of exchange rate variance significantly affect the utilities and portfolio choice of both domestic and foreign investors Jorion (1995), Hull (1987). As a result, the variance and correlations in returns are unpredictable. There is a need to hedge risk against any unforeseen circumstances. In this model, φ represents any sudden shocks from the economy and political decision and policies, while I represents insurance cover such as retirement savings, pensions, gratuities and large financial institutions that could stand as a cover for investors in the market Johnson (1987). We defined the family of admissible portfolios as $\mathcal{A}_{\mathbb{N}}$ and:

Definition 4.4.1

1. $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is cáglád and $\mathbb{N}$-adapted

2. for all $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$.

$$H[\int_0^{\xi} |\vartheta(\tau) - \lambda(\tau)||I - \varphi\beta_i(\tau)\gamma_i(\tau)| + \omega^2(\tau)(I - \varphi\beta_i(\tau)\gamma_i(\tau))^2]d\tau < \infty$$

3. $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$,then $(I - \varphi\beta_i\gamma_i)\,\omega$ is forward integrable and cáglád with respect to D

4. for all $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$, we have $0 < H[\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)] < \infty$ where $\eta'(x) = \frac{d}{dx}\eta(x)$.

5. for all $I - \varphi\beta_i\gamma_i, \varpi \in \mathcal{A}_{\mathbb{N}}$, there exists $\upsilon > 0$, with ϖ bounded, then $I - \varphi\beta_i\gamma_i + j\varpi \in \mathcal{A}_{\mathbb{N}}$ for all $j \in (-\upsilon, \upsilon)$ the equation

$$\eta'(X_{I-\varphi\beta_i\gamma_i+j\varpi}(\xi))X_{I-\varphi\beta_i\gamma_i+j\varpi}(\xi)|N_{I-\varphi\beta_i\gamma_i+j\varpi}(\xi)|_{j\in(-\upsilon,\upsilon)}$$

is uniformly integrable, where

$$N_{I-\varphi\beta_i\gamma_i}(\tau) := \int_0^{\tau} [\vartheta(s) - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)]ds + \int_0^{\tau} \omega(s)dD(s)$$

for $\tau \in [0, \xi]$

6. A buy-hold sell strategy ϖ, that is,

$$\varpi(t) = \alpha I(\tau, \tau + f](t), \quad t \in [0, \xi]$$

with $0 \le \tau < \tau + f \le \xi$ and α $\mathcal{N}_t$-measurable, belonging to $\mathcal{A}_{\mathbb{N}}$.

Then, the portfolio $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal if

$$H[\eta(X_{I-\varphi\beta_i\gamma_i+j\varpi}(\xi))] = H[\eta(X_{\nu_1}(\xi))]$$

for all $\varpi \in \mathcal{A}_{\mathbb{N}}$ bounded and $j \in (-\upsilon, \upsilon)$ with $\upsilon > 0$ given in (5).

Definition 4.4.2

Suppose ϖ is a forward integrable stochastic process and N random variable. Then, the product $N\varpi$ is forward integrable stochastic process and

$$\int_0^\xi N\varpi(t)d^-D(t) = N\int_0^\xi \varpi(t)dD^-D(t) \tag{4.36}$$

where $\varpi = X(t)\nu(t)^*$ such that $\nu_1^* = I - \varphi\beta_i\gamma_i$. Firstly, supposing $I - \varphi\beta_i\gamma_i$ is optimal. Then, for all $\varpi \in \mathcal{A}_{\mathbb{N}}$ bounded implies

$$0 = \frac{d}{dj}H[\eta(X_{I-\varphi\beta_i\gamma_i} + j\varpi(\xi))]|_{j=0}$$

$$\begin{aligned} 0 = H[\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)\int_0^\xi \varpi(s)[\vartheta - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)]ds \\ + \int_0^\xi \varpi(s)\omega(s)d^-D(s)] \end{aligned} \tag{4.37}$$

Now fix $\tau, f : 0 \leq \tau < \tau+f \leq \xi$ and choose $\varpi(s) = \alpha I(\tau, \tau+f](\tau),\ \tau \in [0,\xi]$, where α is a bounded and arbitrary $\mathcal{N}_\tau$-measurable random variable. (4.45) becomes

$$\begin{aligned} 0 = H[\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)\int_\tau^{\tau+f} [\vartheta - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)]ds \\ + \int_\tau^{\tau+f} \omega(s)d^-D(s)\alpha] \end{aligned} \tag{4.38}$$

since it holds for all α,we conclude, that

$$H[F_{I-\varphi\beta_i\gamma_i}(\xi)(N_{I-\varphi\beta_i\gamma_i}(\tau+f) - N_{I-\varphi\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}] = 0$$

where

$$F_{I-\varphi\beta_i\gamma_i}(\xi) = \frac{\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_1\gamma_1}(\xi)}{H[\eta(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)]}$$

and

$$\begin{aligned} N_{I-\varphi\beta_i\gamma_i}(\tau) := \exp\int_0^\tau [\vartheta - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)]ds] \\ + \int_0^\tau [\omega(s)d^-Ds],\ \tau \in [0,\xi] \end{aligned} \tag{4.39}$$

that is,

$$\begin{aligned} HF_{I-\varphi\beta_i\gamma_i}(\xi)\int_\tau^{\tau+f} \vartheta(s) - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)ds \\ + \int_\tau^{\tau+f} \omega(s)d^-D(s)|_{\mathcal{N}_\tau} = 0 \end{aligned} \tag{4.40}$$

by application of Bayes Theorem,

$$\begin{aligned} & H_{Q_{I-\varphi\beta_i\gamma_i}}[N_{I-\varphi\beta_i\gamma_i}(t+f) - N_{I-\varphi\beta_i\gamma_i}(t)|_{\mathcal{N}_t}] \\ 0 = H[F_{I-\varphi\beta_i\gamma_i}(\xi)|_{\mathcal{N}_t}]^{-} & H[F_{I-\varphi\beta_i\gamma_i}(\xi)(N_{I-\varphi\beta_i\gamma_i}(t+f) - N_{I-\varphi\beta_i\gamma_i}(t))|_{\mathcal{N}_t}] \end{aligned} \tag{4.41}$$

since $N_{I-\varphi\beta_i\gamma_i}(t)$ is $\mathcal{N}_t$-adapted, this gives

$$H_{Q_{I-\varphi\beta_i\gamma_i}}[N_{I-\varphi\beta_i\gamma_i}(t+f)|_{\mathcal{N}_t}] = N_{I-\varphi\beta_i\gamma_i}(t)$$

Hence, $N_{I-\varphi\beta_i\gamma_i}(t)$ is an $\mathcal{N}_t, Q_{I-\varphi\beta_i\gamma_i}$-martingale.
Let the probability measure $Q_{I-\varphi\beta_i\gamma_i}$ on $\mathcal{N}_t$ be

$$dQ_{I-\varphi\beta_i\gamma_i} = F_{I-\varphi\beta_i\gamma_i}(\xi)dm$$

and set $H_{Q_{I-\varphi\beta_i\gamma_i}}(t)$ to represent

$Q_{I-\varphi\beta_i\gamma_i}$ expectation then

$$H[F_{I-\varphi\beta_i\gamma_i}(\xi)(N_{I-\varphi\beta_i\gamma_i}(t+f) - N_{I-\varphi\beta_i\gamma_i}(t))|_{\mathcal{N}_t}] = 0$$

written as

$$H_{Q_{I-\varphi\beta_i\gamma_i}}[N_{I-\varphi\beta_i\gamma_i}(t+f) - N_{I-\varphi\beta_i\gamma_i}(t)|_{\mathcal{N}_t}] = 0$$

therefore, $N_{I-\varphi\beta_i\gamma_i}(t),\ t \in [0,t]$, is an

$(\mathbb{N}, Q_{I-\varphi\beta_i\gamma_i})$-martingale that is a martingale with respect to the filtration $\mathbb{N}$ under the probability measure. $Q_{I-\varphi\beta_i\gamma_i}$ Conversely, assuming $N_{I-\varphi\beta_i\gamma_i}$ is an $\mathbb{N}, Q_{I-\varphi\beta_i\gamma_i}$-martingale, then

$$H_{Q_{I-\varphi\beta_i\gamma_i}}[N_{I-\varphi\beta_i\gamma_i}(s+f) - N_{I-\varphi\beta_i\gamma_i}(s)_{\mathcal{N}_s}] = 0$$

for all τ, f therefore $0 \leq \tau < \tau + f \leq \xi$. Equivalently,

$$H_{Q_{I-\varphi\beta_i\gamma_i}}[N_{I-\varphi\beta_i\gamma_i}(\tau+f) - N_{I-\varphi\beta_i\gamma_i}(\tau)\alpha] = 0$$

for all α bounded $\mathcal{N}_t$-measurable. Then,

$$\begin{aligned} 0 = H[\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi) & \int_\tau^{\tau+f} [\vartheta - \lambda(s) - \omega^2(s)I - \beta_i\gamma_i(s)]dt \\ & + \int_\tau^{\tau+f} \omega^2(s)d^{-}D(s)\alpha] \end{aligned} \tag{4.42}$$

holds. taking linear combination, it is valid for all step processes $\varpi \in \mathcal{A}_{\mathbb{N}}$ of cáglád. Referencing assumption (1) and (5)

$$0 = H[\eta'(I - X_{\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)\int_0^\xi \varpi(s)[\vartheta - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)]ds + \int_0^\xi \varpi(s)\omega(s)d^-D(s)] \tag{4.43}$$

holds with boundedness of all $\varpi \in \mathcal{A}_{\mathbb{N}}$. Since

$$j \to H[\eta(X_{I-\varphi\beta_i\gamma_i+j\varpi}(\xi))],\ j \in (-\nu, \nu)$$

maximum is obtain at $j = 0$. Thus,

$$0 = \frac{d}{dj}H[U(X_{I-\varphi\beta_i\gamma_i+j\varpi}(\xi))]|_{j=0}$$

Definition 4.4.3

$I - \beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal in relation with the equation

$$\sup_{I-\varphi\beta_i\gamma_i\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{I-\varphi\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{I-\varphi\beta_i\gamma_i}(\xi))]$$

if

$$N_{I-\varphi\beta_i\gamma_i}(s) := \exp\int_0^s [\vartheta - \lambda(s) - \omega^2(s)I - \varphi\beta_i\gamma_i(s)]ds + \int_0^s \omega(t)d^-D(s)] \tag{4.44}$$

is a $(\mathbb{N}, Q_{I-\varphi\beta_i\gamma_i})$-martingale. Conversely, it is also stated as follows

Theorem 4.4.1

1. $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal for the equation below

$$\sup_{I-\varphi\beta_i\gamma_i\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{I-\varphi\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{I-\varphi\beta_i\gamma_i}(\xi))]$$

only if

$$\hat{N}_{I-\varphi\beta_i\gamma_i}(s) := N_{I-\varphi\beta_i\gamma_i}(s) - \int_0^s \frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma_{I-\varphi\beta_i\gamma_i}](\tau)}{\varsigma_{I-\varphi\beta_i\gamma_i}(\tau)},\ s \in [0, \xi]$$

is a $(\mathbb{N}, m)$-martingale. In this regard,

$$\varsigma(s) := H_{Q_{I-\varphi\beta_i\gamma_i}}\left[\frac{dm}{dQ_{I-\varphi\beta_i\gamma_i}}|_{\mathcal{N}_s}\right] = (H[F_{I-\varphi\beta_i\gamma_i}(\xi)|\mathcal{N}_s])^{-1},\ \ s \in [0, \xi]$$

2. However, Assuming an optimal portfolio $I - \varphi\beta_i\gamma_i \in \mathcal{A}$ exists, then

$$Z(s) = \int_0^s \omega(\tau) d^- D(\tau)$$

is a semimartingale of $(\mathcal{N}, m)$

3. Supposing an optimal $I - \varphi\beta_i\gamma_i \in \mathcal{A}$ exists such that $\omega \neq 0$ for

$$a.a(s, w) \in [0, \xi] \times \Omega$$

then, $D(\tau)$ is an $(\mathcal{N}, m)$-semimartingale

Proof

1. If $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal, then by Definition 4.4.3, $N_{I-\varphi\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{I-\varphi\beta_i\gamma_i})$-martingale with

$$F_{I-\varphi\beta_i\gamma_i}(\xi) = \frac{\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_1\gamma_1}(\xi)}{H[\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)]}$$

and (4.47) and by Girsanov, we have

$$\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau) := N_{I-\varphi\beta_i\gamma_i}(\tau) - \int_0^\tau \frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma](s)}{\varsigma(s)}, \ \tau \in [0, \xi]$$

is a $(\mathbb{N}, m)$-martingale with

$$\varsigma(\tau) := H_{Q_{I-\varphi\beta_i\gamma_i}}[\frac{dm}{dQ_{I-\varphi\beta_i\gamma_i}}]|_{\mathcal{N}_\tau} = H[(F_{I-\varphi\beta_i\gamma_i}(\xi))^{-1} \frac{F_{I-\varphi\beta_i\gamma_i}(\xi)}{H[F_{I-\varphi\beta_i\gamma_i}(\xi)|_{\mathcal{N}_\tau}|_{\mathcal{N}_s}]}]$$

$$= (H[F_{I-\varphi\beta_i\gamma_i}(\xi)|\tilde{\mathcal{N}_\tau}])^{-1} s \in [0, \xi]$$

conversely, if $N_{I-\varphi\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{I-\varphi\beta_1\gamma_1})$-martingale
then $N_{I-\varphi\beta_i\gamma_i}$ is a $(\mathbb{N}, m_{I-\varphi\beta_1\gamma_1})$-martingale and hence $I - \varphi\beta_i\gamma_i$ is optimal by Definition 4.4.3

2. is derived from (1)

3. By (2) recall that

$$Z(\tau) = \int_0^\tau \omega(s) d^- D(s)$$

is a $(\mathcal{N}, m)$-semimartingale. Then Assuming $\omega \neq 0$ for

$$a.a(s, D) \in [0, \xi] \times \hat{\Omega}$$

holds, we obtain that

$$\int_0^s \omega^-(\tau)dZ(\tau) = \int_0^s \omega^-(\tau)\omega(\tau)d^-D(\tau) = D(\tau)$$

is a $(\mathcal{N}, m)$-martingale also.

Theorem 4.4.1 indicate a clear connection between the optimal portfolio $I - \varphi\beta_i\gamma_i$ and the decomposition of the semimartingale D with respect to $\mathbb{N}$. we prove this in the context of portfolio diversification.

Theorem 4.4.2

1. Given that $I - \varphi\beta_i\gamma_i$ is optimal for

$$\sup_{I-\varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}} H[\eta(X_{I-\varphi\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{I-\varphi\beta_i\gamma_i}(\xi))].$$

   Then, D is a semimartingale with respect to $\mathbb{N}$ with a decomposition

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)I - \varphi\beta_i\gamma_i(\tau) - \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{I-\varphi\beta_i\gamma_i}, \varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

   where $\hat{D}$ is a $(\mathbb{N}, m)$-Brownian motion

2. in reverse, assume D is a semimartingale with respect to $(\mathbb{N}, m)$ with a decomposition $dD(\tau) = d\hat{D}(\tau) + dA(\tau)$. And $\mathbb{N}$-adapted finite variation process A and α in $\mathbb{N}$. Assuming $\alpha(\tau)d\tau = dA(\tau) =$ i.e. $dA(\tau)$ is absolutely continuous with respect to $d\tau$ then the solution $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ for

$$\omega(\tau)I - \varphi\beta_i\gamma_i + \frac{1}{\omega(\tau)\varsigma(\tau)} \frac{d\left[N_{I-\varphi\beta_i\gamma_i}, \varsigma\right](\tau)}{d\tau}$$

$$= \alpha(\tau) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}.$$

Then, $I - \varphi\beta_i\gamma_i$ is optimal for

$$\sup_{I-\varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}} H[\eta(X_{I-\varphi\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{I-\varphi\beta_i\gamma_i}(\xi))]$$

since the quadratic variation $N_{I-\varphi\beta_i\gamma_i}$ is continuous absolutely.

$$[N_{I-\varphi\beta_i\gamma_i}, N_{I-\varphi\beta_i\gamma_i}](\tau) = \int_0^\tau \omega^2(s)ds$$

from

$$\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau) := N_{I-\varphi\beta_i\gamma_i}(\tau) - \int_0^\tau \frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma](s)}{\varsigma(s)},\ \tau \in [0, \xi]$$

this implies $d[N_{I-\varphi\beta_i\gamma_i}, \varsigma](\tau)$ is continuous absolutely with respect to $d\tau$. As such,

$$\frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma](\tau)}{\omega(\tau)\varsigma(\tau)} = \frac{1}{\omega(\tau)\varsigma(\tau)} \frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma]\tau d\tau}{d\tau}$$

Proof

Assuming $I - \varphi\beta_i\gamma_i$ is optimal, by Theorem 4.4.2, the equation below

$$\omega^{-1}(\tau)d\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau) = dD(\tau) + \omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)I - \varphi\beta_i\gamma_i(\tau))d\tau - \frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma](\tau)}{\varsigma(\tau)}] \tag{4.45}$$

$\tau \in [0, \xi]$ is a $(\mathbb{N}, m)$-martingale. as much as the process of the quadratic variation $\omega^{-1}(\tau)d\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau)$ $\tau \in [0, \xi]$, is τ, $\tau \in [0, \xi]$ it implies that

$$d\hat{D}(\tau) := \tilde{\omega}^{-1}(\tau)d\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau), \quad \tau \in [0, \xi]$$

is a $(\mathbb{N}, m)$-Brownian motion and

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)I - \varphi\beta_i(\tau)\gamma_i(\tau) - \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{I-\varphi\beta_i\gamma_i}\varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

follows

1. Supposing the decomposition of D is a semimartingale of $(\mathbb{N}, m)$

   $$dD(\tau) = d\hat{D}(\tau) + dA(\tau)$$

   with reference to (2). Set ν to be

   $$\nu = \alpha(t) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}.$$

   Then,

   $$\omega^{-1}(\tau)d\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau) = dD(\tau) + \omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)I - \varphi\beta_i(\tau)\gamma_1(\tau))d\tau - \frac{d[N_{I-\varphi\beta_i\gamma_i}, \varsigma](\tau)}{\varsigma(\tau)}] \tag{4.46}$$

   $\tau \in [0, \xi]$

   $$dD(\tau) - dA(\tau) = d\hat{D}(\tau)$$

therefore

$$\omega^{-1}(\tau)d\hat{N}_{I-\varphi\beta_i\gamma_i}(\tau),\ \tau \in [0,\xi],$$

is a $(\mathbb{N}, m)$-martingale in this manner, $I - \varphi\beta_i\gamma_i$

is optimal via Theorem 4.4.2

For example, If

$$\eta(s) = \log(x), \quad x > 0$$

1. We define a^* as

$$\nu^*(s) = \frac{\vartheta(s) - \lambda(s)}{\omega^2(s)} + \frac{a^*(s)}{\omega(s)}$$

and set

$$D(s) = \frac{\vartheta(s) - \lambda(s)}{\omega(s)}.$$

Then,

$$\hat{D}(t) := D(t) - \int_0^t a^*(s)ds$$

is $(\mathcal{N}, m)$ Brownian motion

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}(D(s) + a^*(s))^2\}ds] \tag{4.47}$$

2. Assuming $D(s)$ is $\mathcal{N}_s$-measurable $s \geq 0$. Then,

$$H[\int_0^\xi D(s)a^*(s)ds] = 0$$

and the corresponding value is

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \exp x_0 + H[\int_0^\xi \{\lambda(s) \\ &+\frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds] \end{aligned} \tag{4.48}$$

Proof

In as much as ν^* is admissible, then the corresponding optimal value function

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \log x_0 + H[\int_0^\xi \{\lambda(s) \\ &+\frac{1}{2}(D(s) + a^*(s))^2\}ds] \end{aligned} \tag{4.49}$$

is finite. It remains to prove

$$H\left[\int_0^\xi D(s)a^*(s)ds\right] = 0$$

If $D(s)$ is $\mathcal{N}$-adapted, then $\hat{D}(t) := D(t) - A(t)$

$$\begin{aligned} H\left[\int_0^\xi D(s)a^* ds\right] &= H\left[\int_0^\xi D(s)\left(dD(s) - d\hat{D}(s)\right)\right] \\ &= H\left[\int_0^\xi D(s)dD(s)\right] - H\left[\int_0^\xi D(s)d\hat{D}(s)\right] = 0. \end{aligned} \tag{4.50}$$

From

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds]$$

Let

$$(a^*(s)) = I - \varphi\beta_i\gamma_i.$$

This equally means that our mode of investment can be diversified

$$\frac{1}{2}(I - \varphi\beta_i\gamma_i + D(s)^2) = (I - \varphi\beta_i\gamma_i + D(s))(I - \varphi\beta_i\gamma_i + D(s))$$

$$I - \varphi\beta_i^2\gamma_i^2 + I - \varphi\beta_i\gamma_i B(s) + I - \varphi\beta_i\gamma_i D(s) + D(s)^2$$
$$\frac{1}{2}[I - \varphi\beta_i^2\gamma_i^2 + 2I - \varphi\beta_i\gamma_i D(s) + D(s)^2]$$
$$\frac{1}{2}[D(s)^2 + I - \varphi\beta_i^2\gamma_i^2 + 2I - \varphi\beta_i\gamma_1 D(s)].$$

Hence, from

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[D(s)^2 + (\omega^*(s))^2]\}ds]$$

we have

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log x_0 + H[\int_0^\xi \{\lambda(s) + \frac{1}{2}[(B(s)^2 + (I - \varphi\beta_i^2\gamma_i^2) + (2I - \varphi\beta_i\gamma_i D(s))]\}ds]$$

since ν is admissible, then the similar optimal value function in the previous equation is finite. Consequently if $D(s)$ is $\mathcal{N}_s$-adapted, $0 \le t \le \xi$. Then,

$$H[\int_0^\xi I - \varphi\beta_i\gamma_i D(s)ds] = 0$$

By $\hat{D}(t) := D(t) - A(t)$ being an $\mathcal{N}_t$-Brownian motion, therefore,

$$H[\int_0^\xi D(s)I - \varphi\beta_i\gamma_i ds] = H[\int_0^\xi D(s)\left(dD(s) - d\hat{D}(s)\right)]$$

$$= H[\int_0^\xi D(s)dD(s)] - H[\int_0^\xi D(s)d\hat{D}(s)] = 0$$

1. Given $\eta(x) = \frac{1}{h}x^h, \;\; x > 0$ where $h \in (0,1)$ we have

$$\eta'(X_{I-\varphi\beta_i\gamma_i+h\varpi}(\xi))X_{I-\varphi\beta_i\gamma_i+h\varpi}(\xi)|M(h)| = X_{I-\varphi\beta_i\gamma_i^h+h\varpi}(\xi)|M(h)$$

and condition (4) in our earlier Definition is satisfied if

$$\sup_{h\in(-\delta,\delta)} H[(X^h_{I-\varphi\beta_i\gamma_i} + h\varpi(\xi)|M(h)|)^{\hat{p}}] < \infty$$

for $\hat{p} > 1$ then set

$$X_{I-\varphi\beta_i\gamma_i+h\varpi}(\xi) = X_{I-\varphi\beta_i\gamma_i}(\xi)N(h).$$

From the Holders inequality,

$$H[(X^h_{I-\varphi\beta_i\gamma_i+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{I-\varphi\beta_i\gamma_i}(\xi))^{h\hat{p}\tilde{a}_1\tilde{b}_1}])^{\frac{1}{\tilde{a}_1\tilde{b}_1}}$$

$$(H[(N(h))^{h\hat{p}\tilde{a}_1\tilde{b}_2}])^{\frac{1}{\tilde{a}_1\tilde{b}_2}}(H[(|M(h)|)^{\hat{p}\tilde{a}_2}])^{\frac{1}{\tilde{a}_2}}$$

where $\tilde{a}_1, \tilde{a}_2 : \frac{1}{\tilde{a}_1} + \frac{1}{\tilde{a}_2} = 1$ and $\tilde{b}_1, \tilde{b}_2 : \frac{1}{\tilde{b}_1} + \frac{1}{\tilde{b}_2} = 1$ Choosing $\tilde{a}_1 = \frac{2}{2-\hat{p}}$, $\tilde{a}_2 = \frac{2}{\hat{p}}$ and also $\tilde{b}_1 = \frac{2-\hat{p}}{h\hat{p}}$, $\tilde{b}_2 = \frac{2-\hat{p}}{2-\hat{p}-h\hat{p}}$ for some $\hat{p} \in (1, \frac{2}{h+1})$. Hence,

$$H[(X^h_{I-\varphi\beta_i\gamma_i+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{I-\varphi\beta_i\gamma_i}(\xi))^2])^{\frac{h\hat{p}}{2}}$$

$$(H[(N(h))^{\frac{2h\hat{p}}{2-\hat{p}-h\hat{p}}}])^{\frac{2-\hat{p}-h\hat{p}}{2}}(H[(|M(h)|^2)])^{\frac{\hat{p}}{2}}$$

supposing $N_{I-\varphi\beta_i\gamma_i}(\xi)$ in

$$\begin{aligned} N(t) = x\exp\{\int_0^s &[\lambda(\tau) + (\varsigma(\tau) - \lambda(\tau))\beta_i(\tau)\gamma_i(\tau) \\ -\frac{1}{2}\omega^2(\tau)(I - \varphi\beta_i(\tau)\gamma_i(\tau))^2]d\tau &+ \int_0^s \omega(\tau)I - \varphi\beta_i(\tau)\gamma_i(\tau)d^- D(\tau) \end{aligned} \tag{4.51}$$

satisfies

$$H[(N_{I-\varphi\beta_i\gamma_i}(\xi))^2] < \infty.$$

Then, (4) and (5) of our earlier definitions is valid if

$$\sup_{h\in-\delta,\delta} H[(\hat{N}(h))\frac{2h\hat{p}}{2-\hat{p}-x\hat{p}}] < \infty$$

however

$$\sup_{h\in-\delta,\delta} H[(\hat{N}(h))\frac{2h\hat{p}}{2-\hat{p}-x\hat{p}}] < \infty$$

holds if for example

$$H\exp\{k\int_0^s [|\vartheta(\tau) - \lambda(\tau)| + |I - \varphi\beta_i\gamma_i(\tau)|d\tau\}] < \infty \;\forall\; k > 0$$

however,

$$H[(\hat{N}_{I-\varphi\beta_i\gamma_i}(\xi))^2] < \infty$$

is equally verify for $k > 0$

$$\begin{aligned} H\exp\{k(\int_0^{\xi}[|\vartheta(\tau)-\lambda(\tau)|+|I-\varphi\beta_i\gamma_i(\tau)|d\tau])\} \\ +|\int_0^{\xi} I-\varphi\beta_i\gamma_i(\tau)\omega(\tau)dD(\tau)| < \infty \end{aligned} \tag{4.52}$$

4.5 Assets diversification for large scale investors under insurance cover

A sensitive investor spread his investment network on different assets to reduce risk and more return on investment. we defined

$$H : \mathbb{R}^n \to \langle\beta,\gamma\rangle, \subseteq \mathbb{R}$$

where

$$\langle\hat{\beta,}\hat{\gamma}\rangle = \langle(\beta_1\beta_2),(\gamma_1\gamma_2),\cdots\rangle = \sum_{i=1}^{n} I-\beta_i\gamma_i$$

where $\sum_{i=1}^{n}(I-\nu_i\beta_i\gamma_i)$ and ν_i is a confine placed before investor's on the size of assets to be held due to costs of transactions. $\nu_i = 1$ if assets i is chosen in the portfolio, and 0 otherwise. Large investor's engage in effective portfolio management, paying close attention to market trends against spontaneous shift in the economy and changes to the political landscape as well as factors that may affect some organisations . This is to enable efficient monitoring of transactions of assets, the risk of exchange rate variance significantly affect the portfolio choice of both domestic and foreign investors as stated by Jorion (1995)and Hull (1987). As a result, the variance and correlations in returns are unpredictable. Therefore, there is a need to hedge risk against any unforeseen circumstances. In this model, φ represents any sudden shocks from the economy and political decision and policies, while I represents insurance cover such as retirement savings, pensions, gratuities and large financial institutions for a backup Johnson (1987).

Definition 4.5.1

$\mathcal{A}_{\mathbb{N}}$ is admissible portfolios expressed as

1. all $\sum_{i=1}^{n}(I-\varphi\beta_i\gamma_i)) \in \mathcal{A}_{\mathbb{N}}$ is c$\acute{a}$gl$\acute{a}$d and $\mathbb{N}$-adapted

2. $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$.

$$H[\int_0^{\xi} |\vartheta(\tau) - \lambda(\tau)|| (I - \varphi\beta_i(\tau)\gamma_i(\tau)|) + \sum_{i=1}^{n} ((I - v\varphi\beta_i(\tau)\gamma_1(\tau))^2]d\tau < \infty$$

3. $I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$,the product $(\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i)\,\omega$ is forward integrable and *cáglád*

4. $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$, then $0 < H[\eta'(L_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))L_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)] < \infty$ where $\eta'(l) = \frac{d}{dl}\eta(l)$.

5. for all $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i, \varpi \in \mathcal{A}_{\mathbb{N}}$, there exist $\delta > 0$, with a bounded ϖ, such that $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + j\varpi \in \mathcal{A}_{\mathbb{N}}$ in all $j \in (-\delta, \delta)$ as such, the entire family

$$\eta'(L_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i+j\varpi}(\xi))L_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i+j\varpi}(\xi)|N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i+j\varpi}(\xi)|_{j\in(-\delta,\delta)}$$

is uniformly integrable, where

$$N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) := \int_0^{s} [\lambda(\tau)-\lambda(\tau)-\omega^2(\tau)\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i(\tau)]d\tau + \int_0^{s} \omega(s)dD(\tau),$$

6. ϖ portfolio permits buy-low and sell-high strategy, i.e.

$$\varpi(s) = \alpha I(\tau, \tau + f](t), \quad t \in [0, \xi]$$

with $0 \leq \tau < s + f \leq \xi$ and α $\mathcal{N}_{\tau}$-measurable belonging to $\mathcal{A}_{\mathbb{N}}$. Then, $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal if

$$H[\eta(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i+j\varpi}(\xi))] = H[\eta(X_{\nu_1}(\xi))]$$

for a bounded $\varpi \in \mathcal{A}_{\mathbb{N}}$ and $y \in (-\delta, \delta)$ with $\delta > 0$ given in (5)of definition (4.5.1).

Definition 4.5.2

Assuming ϖ is a forward integrable stochastic process and N a random variable, then the product $N\varpi$ is a stochastic process and forward integrable thus,

$$\int_0^{\xi} N\varpi(\tau)d^{-}D(\tau) = N\int_0^{\xi} \varpi(\tau)d^{-}D(\tau)$$

where $\varpi = X(\tau)\nu(\tau)^*$ in this sense, $\nu_1^* = \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i$. Firstly, $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i$ is optimal. For a bounded $\varpi \in \mathcal{A}_{\mathbb{N}}$ we have

$$0 = \frac{d}{dj}H[\eta(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i+j\varpi}(\xi))]|_{j=0}$$

$$0 = H[\eta'(X(\xi))X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)$$
$$\int_0^\xi \varpi(\tau)[\vartheta - \lambda(\tau) - \omega^2(s)\sum_{i=1}^n I - \varphi\beta_i\gamma_i(\tau)]d\tau \tag{4.53}$$
$$+\int_0^\xi \varpi(\tau)\omega(\tau)d^- D(\tau)]$$

fixing $\tau, f : 0 \le \tau < \tau + \tau \le \xi$ and choosing $\varpi(s) = \alpha I(\tau, \tau + f](s),\ t \in [0, \xi]$, for any stochastic amount α of $\mathcal{N}_\tau$ bounded. equation (4.53) becomes

$$0 = H[\eta'(X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)$$
$$\int_\tau^{\tau+f} [\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^n I - \varphi\beta_i\gamma_i(s)]ds \tag{4.54}$$
$$+\int_\tau^{\tau+f} \omega(s)d^- D(s)\alpha]$$

this holds in all α, then

$$H[F_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)(N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau + f) - N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}] = 0$$

where

$$F_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi) = \frac{\eta'(X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)}{H[\eta'(X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)]}$$

and

$$N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau) := \exp\int_0^\tau [\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^n I - \varphi\beta_i\gamma_i(s)]ds$$
$$+\int_0^\tau \omega(s)d^- D(s)] \tag{4.55}$$

that is,

$$0 = H[F_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)\int_s^{s+f} \alpha(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^n I - \varphi\beta_i\gamma_i(\tau)d\tau$$
$$+\int_s^{s+f} \omega(\tau)d^- D(\tau)|_{\mathcal{N}_\tau}] \tag{4.56}$$

$$H_{Q_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}}[N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau + f) - N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau)|_{\mathcal{N}_\tau}]$$

$$0 = H[F_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)|_{\mathcal{N}_\tau}]^- H[F_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)(N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau + f)$$
$$-N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}] \tag{4.57}$$

since $N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau)$ is $\mathcal{N}_\tau$-adapted, this gives

$$H_{Q_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}}[N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau + f)|_{\mathcal{N}_\tau}] = N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\tau).$$

Hence, $N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau)$ is an $(\mathcal{N}_\tau, Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i)}$-martingale.

Let the probability measure $Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}$ on $\mathcal{N}_\tau$ be

$$dQ_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i} = F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)dm$$

and $H_{Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}}(\tau)$ an expectation of $Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}$ So,

$$H[F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)(N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau+f) - N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau))|_{\mathcal{N}_\tau}] = 0$$

written as

$$H_{Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}}[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau+f) - N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau)|_{\mathcal{N}_\tau}] = 0$$

$N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau),\ \tau \in [0,\xi]$, is an $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i})$-martingale of the filtration $\mathbb{N}$ under $Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}$ measure it can equally be stated as follows that suppose $N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i})$-martingale, then

$$H_{Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}}[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau+f) - N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau)_{\mathcal{N}_\tau}] = 0$$

for all τ, f and then $0 \leq \tau < \tau + f \leq \xi$. Similarly,

$$H_{Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}}[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau+f) - N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau)\alpha] = 0$$

for α $\mathcal{N}_\tau$-measurable. Therefore,

$$\begin{aligned} 0 = H[\eta'(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi) \\ \int_{\tau}^{\tau+f}[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i(s)]ds \\ + \int_{\tau}^{\tau+f} \omega(s)d^{-}D(s)\alpha] \end{aligned} \tag{4.58}$$

holds. taking linear combination

$$\begin{aligned} 0 = H[\eta'(X_{I-\varphi\beta_i\gamma_i}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi) \\ \int_{0}^{\xi}\varpi(s)[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i(s)]ds \\ + \int_{0}^{\xi} \varpi(s)\omega(s)d^{-}D(s)] \end{aligned} \tag{4.59}$$

remain valid for all *cáglád* step processes $\varpi \in \mathcal{A}_{\mathbb{N}}$ from assumption (1) and (5) we have (4.59) still holds for a bounded $\varpi \in \mathcal{A}_{\mathbb{N}}$. Provided the function

$$y \to H[\eta(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i+j\varpi}(\xi))],\ j \in (-\delta, \delta)$$

maximum is attained at $j = 0$. Thus,

$$0 = \frac{d}{dj} H[U(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i + j\varpi}(\xi))]|_{j=0}$$

Definition 4.5.3

A stochastic process $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal as far as the stochastic process

$$N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) := \exp \int_0^{\tau} [\vartheta - \lambda(s) - \omega^2(s) \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i(s)]ds + \int_0^{\tau} \omega(s) d^- D(s)], \ \tau \in [0, \xi] \tag{4.60}$$

is a $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i})$-martingale. By the application of Girsanov's theorem, it is equally stated as follows

Theorem 4.5.1

1. The process $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ for

   $$\sup_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}} H[\eta(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))]$$

   is optimal if

   $$\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) := N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) - \int_0^{\tau} \frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \upsilon_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}](s)}{\upsilon_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(s)}$$

   for $\tau \in [0, \xi]$ is a $(\mathbb{N}, m)$-martingale. In that sense,

   $$\begin{aligned}\upsilon_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) &:= H_{Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}} \left[\frac{dm}{dQ_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}} |_{\mathcal{N}_\tau} \right] \\ &= (H[F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)|\mathcal{N}_\tau])^{-}1\end{aligned}$$

   $t \in [0, \xi]$

2. However, if an optimal portfolio $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}$ exists, hence the process

   $$\varsigma(s) = \int_0^{\tau} \omega(s) d^- D(s)$$

   is a $(\mathcal{N}, m)$-semimartingale

3. supposed $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}$ of optimal value exists and $\omega \neq 0$ for

   $$a.a(s, w) \in [0, \xi] \times \Omega$$

   then $D(\tau)$ is a $(\mathbb{N}, m)$-semimartingale

Proof

1. If $\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ is optimal, by Definition 4.5.3, then $N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}$ is a $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i})$-martingale with

$$F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi) = \frac{\eta'(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)}{H[\eta'(X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)]}$$

and (4.55) respectively, By applying theorem of Girsanov,

$$\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) := N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) - \int_0^\tau \frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma](s)}{\varsigma(s)}$$

is a martingale with respect to $(\mathbb{N}, m)$

$$\begin{aligned} \varsigma(\tau) &:= H_{Q_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}}[\frac{dm}{dQ_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}}]|_{\mathcal{N}_\tau} \\ &= H[(F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))^{-1}\frac{F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)}{H[F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)|_{\mathcal{N}_\tau}|_{\mathcal{N}_\tau}]}] \\ &= (H[F_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi)|\mathcal{N}_t])^{-1}, \tau \in [0,\xi] \end{aligned} \tag{4.61}$$

conversely, if $N_{\sum_{i=1}^{n} I-\varphi\beta_1\gamma_1}$ is a $(\mathbb{N}, Q_{I-\varphi\beta_i\gamma_i})$-martingale, then $N_{\sum_{i=1}^{n} I-\varphi\beta_1\gamma_1}$ is a $(\mathbb{N}, m_{I-\varphi\beta_i\gamma_i})$-martingale and hence $\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i$ is optimal by Definition 4.5.3

2. is derived from (1)

3. By (2) its obvious that

$$Y(t) = \int_0^t \omega(\tau)d^- D(\tau)$$

is a $(\mathbb{N}, m)$-semimartingale. supposing $\omega \neq 0$ for

$$a.a(s, w) \in [0,\xi] \times \Omega$$

its valid, then

$$\int_0^s \omega^-(\tau)dY(\tau) = \int_0^s \omega^-(\tau)\omega(\tau)d^- D(\tau) = D(\tau)$$

is an $(, m)$-semimartingale also.

Theorem 4.5.1 furnishes a clear connection between $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i$ optimal portfolio and decomposition of semimartingale D with respect to $\mathbb{N}$. we prove this in the context of portfolio diversification.

Theorem 4.5.2

1. Given that $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i$ is optimal, then the decomposition of semimartingale D with respect to $\mathbb{N}$ is

   $$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i(\tau) - \frac{\vartheta(\tau) - \lambda(\tau)}{\tilde{\omega}(\tau)}]d\tau + \frac{[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma](\tau)}{\tilde{\omega}\tau\varsigma(\tau)}$$

   where $\hat{D}$ is a $(\mathbb{N}, m)$-Brownian motion

2. in reverse, assume the semimartingale D with respect to $(\mathbb{N}, m)$ decomposes as $dD(\tau) = d\hat{D}(\tau) + dA(\tau)$. where $\hat{D}$ is $(\mathbb{N}, m)$-Brownian motion and $\mathbb{N}$-adapted A finite variation process. Assume $dA(\tau) = \alpha(\tau)d(\tau)$ and for $\mathbb{N}$-adapted process α that is, $dA(\tau)$ is absolutely continuous with respect to $d(\tau)$ then there is a solution $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}$ of the form

   $$\omega(\tau)\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + \frac{1}{\omega(\tau)\varsigma(\tau)}\frac{d\left[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma\right](\tau)}{d\tau}$$

   $$= \alpha(\tau) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}$$

in this sense $\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i$ proves optimal for

$$\sup_{I-\varphi\beta_i\gamma_i \in \mathcal{A}_{\mathbb{N}}} H[\eta(L_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))] = H[\eta(X^*_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\xi))]$$

since quadratic variation of $\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}$ is absolutely continuous that is,

$$[\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}](\tau) = \int_0^{\tau} \omega^2(s)ds$$

from

$$\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) := N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) - \int_0^{\tau} \frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma](s)}{\varsigma(s)}, \tau \in [0, \xi]$$

then $d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma](\tau)$ is absolutely continuous with respect to $d\tau$.

$$\frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma](\tau)}{\vec{\omega}(\tau)\varsigma(\tau)} = \frac{1}{\vec{\omega}(\tau)\varsigma(\tau)}\frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}, \varsigma](\tau)d\tau}{d\tau}$$

Proof

Assuming $\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i$ is optimal, by Theorem 4.5.2,

$$\begin{aligned}\tilde{\omega}^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) &= dD(\tau)\\ +\omega^{-1}(\tau)[(\vartheta(\tau)-\lambda(\tau)-\omega^2(\tau)\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i(\tau))d\tau\\ -\frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i},\varsigma](\tau)}{\varsigma(\tau)}]\end{aligned} \tag{4.62}$$

$\tau\in[0,\xi]$ is a $(\mathbb{N},m)$-martingale and $\omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau)$ $\tau\in[0,\xi]$, then

$$d\hat{D}(\tau) := \omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau),\quad \tau\in[0,\xi]$$

is a $(\mathbb{N},m)$-Brownian motion and

$$dD(\tau) = d\hat{D}(\tau)+[\omega(\tau)\sum_{i=1}^{n} I-\varphi\beta_i(\tau)\gamma_i(\tau)-\frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}]d\tau+\frac{[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i},\varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

holds

1. Assume D is a $(\mathbb{N},m)$-martingale with a decomposition

   $$dD(\tau) = d\hat{D}(\tau)+dA(\tau)$$

   with referencing to (2). Set ν

   $$\nu = \alpha(\tau)+\frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}.$$

   Then,

   $$\begin{aligned}\omega^{-1}(t)d\hat{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau) &= dD(\tau)+\omega^{-1}(t)[(\vartheta(\tau)-\lambda(\tau)\\ -\omega^2(\tau)\sum_{i=1}^{n} I-\varphi\beta_i(t)\gamma_i(\tau))d\tau\\ -\frac{d[N_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i},\varsigma](\tau)}{\varsigma(\tau)}]\end{aligned} \tag{4.63}$$

   $\tau\in[0,\xi]$

   $$= dD(\tau)-dA(\tau) = d\hat{D}(\tau)$$

   consequently

   $$\omega^{-1}(\tau)d\tilde{N}_{\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i}(\tau),\quad \tau\in[0,\xi],$$

   is a $(\mathbb{N},m)$-martingale then $\sum_{i=1}^{n} I-\varphi\beta_i\gamma_i$ is optimal from Theorem 4.9

For example,

$$U(s) = (x), \quad x > 0$$

1. We define a^* as

$$\nu^*(s) = \frac{\vartheta(s) - \lambda(s)}{\omega^2(s)} + \frac{a^*(s)}{\omega(s)}$$

   and set

$$D(s) = \frac{\vartheta(s) - \lambda(s)}{\omega(s)}.$$

   Then,

$$\hat{D}(\tau) := D(\tau) - \int_0^\tau a^*(s)ds$$

   is a $(\mathcal{N}, m)$-Brownian and

$$H_\xi^{\mathbb{F},\mathbb{N}} = \log \iota_0 + H[\int_0^\xi \{\lambda(\tau) + \frac{1}{2}(D\tau) + a^*(\tau))^2\}d\tau] \tag{4.64}$$

2. Assume that $D(s)$ is $\mathcal{N}_s$-measurable, $\xi \geq 0$. Then

$$H[\int_0^\xi D(s)a^*(s)ds] = 0$$

   and the similar value is

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \log x_0 + H[\int_0^\xi \{\lambda(s) \\ &+ \frac{1}{2}[D(s)^2 + (a^*(s))^2]\}ds] \end{aligned} \tag{4.65}$$

Proof

Since ν^* is admissible, then the corresponding optimal value function

$$\begin{aligned} H_\xi^{\mathbb{F},\mathbb{N}} &= \log x_0 + H[\int_0^\xi \{\lambda(s) \\ &+ \frac{1}{2}(D(s) + a^*(s))^2\}ds] \end{aligned} \tag{4.66}$$

is finite. We prove

$$H\left[\int_0^\xi D(\tau)a^*(\tau)d\tau\right] = 0$$

by saying that if $D(s)$ is $\mathcal{F}$-adapted, then by $\hat{D}(t) := D(t) - A(t)$

$$\begin{aligned} H\left[\int D(s)a^*ds\right] &= H\left[\int_0^\xi D(s)Dd(s) - d\hat{D}(s))\right] \\ &= H\left[\int_0^\xi D(s)dD(s)\right] - H\left[\int_0^\xi D(s)d\hat{D}(s)\right] = 0. \end{aligned} \tag{4.67}$$

From

$$H_{\xi}^{\mathbb{F},\mathbb{N}} = \log X_0 + H[\int_0^{\xi} \{\lambda(\tau) + \frac{1}{2}[D(\tau)^2 + (a^*(\tau))^2]\} d\tau]$$

Let

$$(a^*(s)) = \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i.$$

This equally means that our mode of investment can be diversified

$$\frac{1}{2}(\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + D(s))^2 = (\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + D(s))(\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + D(s))$$

$$\sum_{i=1}^{\tilde{n}} I - \varphi\beta_i^2\gamma_i^2 + \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i D(s) + \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i \check{D}(s) + D(s)^2$$

Hence, from

$$H_{\xi}^{\mathbb{F},\mathbb{N}} = \log X_0 + H[\int_0^{\xi} \{\lambda(s) + \frac{1}{2}[D(s)^2 + (a^*(s))^2]\} ds]$$

we have

$$H_{\xi}^{\mathbb{F},\mathbb{N}} = \log x_0 + H \int_0^{\xi} \{\lambda(s) + \frac{1}{2}[(D(s)^2 + (\sum_{i=1}^{n} I - \varphi\beta_1^2\gamma_1^2)\}$$

$$\{+ \sum_{i=1}^{n} I - 2(\varphi\beta_i\gamma_i D(s))\} ds]$$

is finite. As a result, $D(s)$ is $\mathcal{N}_s$-measurable, $0 \leq t \leq \xi$. Therefore, $H[\int_0^{\xi} \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i D(s) ds] = 0$ By $\hat{D}(\tau) := D(\tau) - A(\tau)$ being $\mathcal{N}_t$-Brownian motion, we have

$$H[\int_0^{\xi} D(s) \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i ds] = H[\int_0^{\xi} D(s) \left(dD(s) - d\hat{D}(s)\right)]$$

$$= H[\int_0^{\xi} D(s) dD(s)] - H[\int_0^{\xi} D(s) d\hat{D}(s)] = 0$$

1. Given $\eta(x) = \frac{1}{h}x^h, \;\; x > 0$

where $h \in (0, 1)$ we have

$$\eta'(X_{\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + h\varpi}(\xi)) X_{\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + h\varpi}(\xi) |M(h)| = X_{\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i^h + h\varpi}(\xi) |M(h)$$

and condition (4) in our earlier Definition is satisfied if

$$\sup_{h \in (-\delta, \delta)} H[(X^h_{\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i + h\varpi}(\xi)|M(h)|)^p] < \infty$$

for $p > 1$ set

$$X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i+h\varpi}(\xi) = X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)N(h)$$

where

$$N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(h) := \exp\int_0^\xi [\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^n I - \varphi\beta_i\gamma_i(\tau)]d\tau + \int_0^\xi \omega)dD(\tau).$$

From the Holders inequality, we have

$$H[(X^h_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))^{h\hat{p}a_1b_1}])^{\frac{1}{a_1b_1}}$$

$$(H[(N(h))^{h\hat{p}a_1b_2}])^{\frac{1}{a_1b_2}}(H[(|M(h)|)^{\hat{p}a_2}])^{\frac{1}{a_2}}$$

where $a_1, a_2 : \frac{1}{a_1} + \frac{1}{a_2} = 1$ and $b_1, b_2 : \frac{1}{b_1} + \frac{1}{b_2} = 1$, then we can choose $a_1 = \frac{2}{2-p}$ $a_2 = \frac{2}{\hat{p}}$ and also $b_1 = \frac{2-\hat{p}}{h\hat{p}}$, $b_2 = \frac{2-\hat{p}}{2-\hat{p}-h\hat{p}}$ for some $\hat{p} \in (1, \frac{2}{h+1})$. Hence,

$$H[(X^h_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))^2])^{\frac{hp}{2}}$$

$$(H[(N(h))^{\frac{2h\hat{p}}{2-\hat{p}-h\hat{p}}}])^{\frac{2-\hat{p}-h\hat{p}}{2}}(H[(|M(h)|^2)])^{\frac{\hat{p}}{2}}$$

if $N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi)$ in

$$\begin{aligned} N(s) = x\exp\{\int_0^s \lambda(\tau) + (\vartheta(\tau) - \lambda(\tau))\sum_{i=1}^n I - \varphi\beta_i(\tau)\gamma_i(\tau) \\ -\frac{1}{2}\omega^2(\tau)(\sum_{i=1}^n I - \varphi\beta_i(\tau)\gamma_i(\tau))^2]d\tau + \int_0^s \omega(\tau)\sum_{i=1}^n I - \varphi\beta_i(\tau)\gamma_i(\tau)d^-D(\tau) \end{aligned} \tag{4.68}$$

satisfies

$$H[(N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))^2] < \infty.$$

Then the condition (4) and (5) of our earlier definitions holds if

$$\sup_{h\in-\delta,\delta} H[(N(h))\frac{2hp}{2-p-xp}] < \infty$$

however

$$\sup_{h\in-\delta,\delta} H[(N(h))\frac{2hp}{2-p-xp}] < \infty$$

holds if for example

$$H\exp\{k\int_0^s [|\vartheta(\tau) - \lambda)| + |\sum_{i=1}^n I - \varphi\beta_i\gamma_i(\tau)|d\tau\}] < \infty \;\forall\; k > 0$$

however,

$$H[(N_{\sum_{i=1}^n I-\varphi\beta_i\gamma_i}(\xi))^2] < \infty$$

is equally verify for all $k > 0$

$$H \exp\{k(\int_0^\xi [|\vartheta(\tau) - \lambda(\tau)| + |\sum_{i=1}^{n} I - \varphi\beta_i\gamma_i(\tau)|d\tau)\} \\ +|\int_0^\xi \sum_{i=1}^{n} I - \varphi\beta_i\gamma_i(\tau)\omega(\tau)dD(\tau)| < \infty \tag{4.69}$$

The next section leads us to another interesting part of a sensitive investors optimal portfolio who undergo budgetary constraints with consumption in mind.

4.6 Optimal portfolio of a sensitive investor with budget constraint and consumption

We discuss optimal portfolio of a sensitive investor with consideration of investors consumption and budget constraints. Since in the portfolio framework, the more the consumption, the shorter the portfolio. Therefore, We denote φ as the constraint subject to investors budget and c is the consumption of the investor, while I represents some financial institutions aid where the investor can easily run for aid Øksendal (2006), Mark Schroder (1999) and Samuelson (1969). Thus, we defined $\mathcal{A}_{\mathbb{N}}$ as the set of admissible portfolio for the investor and explicitly expressed in the definition below:

Definition 4.6.1

1. $I - (\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ is cáglád and $\mathbb{N}$-adapted

2. for all $I - (\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$.

   $$H[\int_0^\xi |\vartheta(\tau)-\lambda(\tau)||I-(\varphi\beta_i(\tau)\gamma_i-c)(\tau)|+\omega^2(\tau)(I-(\varphi\beta_i(\tau)\gamma_i(\tau))^2-c)]d\tau < \infty$$

3. $I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}_{\mathbb{N}}$,then $(I - (\varphi\beta_i\gamma_i - c))\,\omega$ is forward integrable and cáglád in respect of D

4. for all $I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}_{\mathbb{N}}$, we have $0 < H[\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)] < \infty$ where $\eta'(x) = \frac{d}{dx}\eta(x)$.

5. for all $I - (\varphi\beta_i\gamma_i - c), \varpi \in \mathcal{A}_{\mathbb{N}}$, there exists $\upsilon > 0$, with ϖ bounded , then $I - (\varphi\beta_i\gamma_i - c) + j\varpi \in \mathcal{A}_{\mathbb{N}}$ for all $j \in (-\upsilon, \upsilon)$ the equation

   $$\eta'(X_{I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi)|N_{I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi)|_{j\in(-\upsilon,\upsilon)}$$

is uniformly integrable, where

$$N_{I-(\varphi\beta_i\gamma_i-c)}(\tau) := \int_0^\tau [\vartheta(s)-\lambda(s)-\omega^2(s)I-(\varphi\beta_i\gamma_i(s)-c)]ds+\int_0^\tau \omega(s)dD(s)$$

for $\tau \in [0,\xi]$

6. A buy-hold sell strategy ϖ, that is

$$\varpi(t) = \alpha I(\tau, \tau+f](t), \;\; t \in [0,\xi]$$

with $0 \leq \tau < \tau + f \leq \xi$ and α $\mathcal{N}_t$-measurable, belonging to $\mathcal{A}_{\mathbb{N}}$. Then the portfolio $I-(\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ is optimal if

$$H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))] = H[\eta(X_{\nu_1}(\xi))]$$

for all $\varpi \in \mathcal{A}_{\mathbb{N}}$ bounded and $j \in (-\upsilon, \upsilon)$ with $\upsilon > 0$ given in (5).

Definition 4.6.2

Assume ϖ is a forward integrable stochastic process and N a random variable, then the product $N\varpi$ is forward integrable stochastic process and implies

$$\int_0^\xi N\varpi(t)d^-D(t) = N\int_0^\xi \varpi(t)dD^-D(t) \tag{4.70}$$

where $\varpi = X(t)\nu(t)^*$ such that $\nu_1^* = I-(\varphi\beta_i\gamma_i - c)$. Firstly, supposing $I-(\varphi\beta_i\gamma_i - c)$ is optimal. Then for all $\varpi \in \mathcal{A}_{\mathbb{N}}$ bounded implies

$$0 = \frac{d}{dj}H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)} + j\varpi(\xi))]|_{j=0}$$

$$\begin{aligned} 0 = H[\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_1-c)}(\xi) \\ \int_0^\xi \varpi(s)[\vartheta-\lambda(s)-\omega^2(s)I-(\varphi\beta_i\gamma_i(s)-c)]ds \\ +\int_0^\xi \varpi(s)\omega(s)d^-D(s)] \end{aligned} \tag{4.71}$$

Now we fix $\tau, f : 0 \leq \tau < \tau + f \leq \xi$ and choose $\varpi(s) = \alpha I(\tau, \tau+f](\tau), \; \tau \in [0,\xi]$, where α is a bounded and arbitrary $\mathcal{N}_\tau$-measurable random variable. Our equation (4.71) becomes

$$\begin{aligned} 0 = H[\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)\int_\tau^{\tau+f}[\vec{\vartheta}-\lambda(s)-\omega^2(s)I-(\varphi\beta_i\gamma_i(s)-c)]ds \\ +\int_\tau^{\tau+f}\omega(s)d^-D(s)\alpha] \end{aligned} \tag{4.72}$$

because it holds for all α, we conclude, that

$$H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)(N_{I-(\varphi\beta_i\gamma_i-c)}(\tau+f)-N_{I-(\varphi\beta_i\gamma_i-c)}(\tau))|_{\mathcal{N}_\tau}]=0$$

where

$$F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)=\frac{\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)}{H[\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)]}$$

and

$$\begin{aligned}N_{I-(\varphi\beta_i\gamma_i-c)}(\tau):=\exp\int_0^\tau[\vartheta-\lambda(s)-\omega^2(s)I-(\varphi\beta_i\gamma_i(s)-c)]ds]\\+\int_0^\tau[\omega(s)d^-Ds],\ \tau\in[0,\xi]\end{aligned}\tag{4.73}$$

that is,

$$\begin{aligned}H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)\int_\tau^{\tau+f}\vartheta(s)-\lambda(s)-\omega^2(s)I-(\varphi\beta_i\gamma_i(s)-c)ds\\+\int_\tau^{\tau+f}\omega(s)d^-D(s)|_{\mathcal{N}_\tau}]=0\end{aligned}\tag{4.74}$$

by application of Bayes Theorem,

$$H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}[N_{I-(\varphi\beta_i\gamma_i-c)}(t+f)-N_{I-(\varphi\beta_i\gamma_i}(t)-c)|_{\mathcal{N}_t}]$$

$$\begin{aligned}0=H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)|_{\mathcal{N}_t}]^-H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)(N_{I-(\varphi\beta_i\gamma_i-c)}(t+f)]\\-N_{I-(\varphi\beta_i\gamma_i}(t)-c))|_{\mathcal{N}_t}\end{aligned}\tag{4.75}$$

since $N_{I-(\varphi\beta_i\gamma_i}(t)-c)$ is $\mathcal{N}_t$-adapted, this gives

$$H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}[N_{I-(\varphi\beta_i\gamma_i-c)}(t+f)|_{\mathcal{N}_t}]=N_{I-(\varphi\beta_i\gamma_i-c)}(t)$$

Hence $N_{I-(\varphi\beta_i\gamma_i}(t)-c)$ is an $\mathcal{N}_t, Q_{I-(\varphi\beta_i\gamma_i-c)}$-martingale.
Let the probability measure $Q_{I-(\varphi\beta_i\gamma_i-c)}$ on $\mathcal{N}_t$ be

$$dQ_{I-(\varphi\beta_i\gamma_i-c)}=F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)dm$$

and set $H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}(t)$
as $Q_{I-(\varphi\beta_i\gamma_i-c)}$ expectation then

$$H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)(N_{I-(\varphi\beta_i\gamma_--c)}(t+f)-N_{I-(\varphi\beta_i\gamma_i-c)}(t))|_{\mathcal{N}_t}]=0$$

written as

$$H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}[N_{I-(\varphi\beta_i\gamma_i-c)}(t+f)-N_{I-(\varphi\beta_i\gamma_i-c)}(t)|_{\mathcal{N}_t}]=0$$

therefore, $N_{I-(\varphi\beta_i\gamma_i-c)}(t),\ t \in [0,t]$, is a

$(\mathbb{N}, Q_{I-(\varphi\beta_i\gamma_i-c)})$-martingale of $\mathbb{N}$ under $Q_{I-(\varphi\beta_i\gamma_i-c)}$ Conversely, assuming $\mathbb{N}, Q_{I-(\varphi\beta_i\gamma_i-c)}$-martingale of $N_{I-(\varphi\beta_i\gamma_i-c)}$ holds, then

$$H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}[N_{I-(\varphi\beta_i\gamma_i-c)}(s+f) - N_{I-(\varphi\beta_i\gamma_i-c)}(s)_{\mathcal{N}_s}] = 0$$

in all τ, f therefore $0 \leq \tau < \tau + f \leq \xi$. Equivalently,

$$H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}[N_{I-(\varphi\beta_i\gamma_i-c)}(\tau+f) - N_{I-(\varphi\beta_i\gamma_i-c)}(\tau)\alpha] = 0$$

for all α bounded $\mathcal{N}_t$-measurable. Then,

$$\begin{aligned} 0 = H[\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-\varphi\beta_i\gamma_i}(\xi)\int_\tau^{\tau+f}[\vartheta - \lambda(s) - \omega^2(s)I - (\varpi\beta_i\gamma_i - c)(s)]dt \\ + \int_\tau^{\tau+f}\omega(s)d^- D(s)\alpha] \end{aligned} \tag{4.76}$$

holds. taking linear combination, it is valid for all step processes $\varpi \in \mathcal{A}_{\mathbb{N}}$ of cáglád. Referencing assumption (1) and (5)

$$\begin{aligned} 0 = H[\eta'(I - X_{(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi) \\ \int_0^\xi \varpi(s)[\vartheta - \lambda(s) - \omega^2(s)I - (\varphi\beta_i\gamma_i - c)(s)]ds \\ + \int_0^\xi \varpi(s)\omega(s)d^- D(s)] \end{aligned} \tag{4.77}$$

holds with boundedness of all $\varpi \in \mathcal{A}_{\mathbb{N}}$. As far as the

$$j \to H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))],\ j \in (-\nu, \nu)$$

maximum is obtain at $j = 0$. Thus,

$$0 = \frac{d}{dj}H[U(X_{I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))]|_{j=0}$$

Definition 4.6.3

$I - \beta_1\gamma_1 \in \mathcal{A}_{\mathbb{N}}$ is optimal in relation with the equation

$$\sup_{I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))] = H[\eta(X^*_{I-(\varphi\beta_i\gamma_i-c)}(\xi))]$$

if

$$\begin{aligned} N_{I-(\varphi\beta_i\gamma_i-c)}(s) := \exp\int_0^s [\vartheta - \lambda(s) - \omega^2(s)I - (\varphi\beta_i\gamma_i - c)(s)]ds \\ + \int_0^s \omega(t)d^- D(s)] \end{aligned} \tag{4.78}$$

is a $(\mathbb{N}, Q_{I-(\varphi\beta_i\gamma_i-c)})$-martingale. Conversely, it is also stated as follows

Theorem 4.6.1

1. $I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}_{\mathbb{N}}$ is optimal for the equation below

$$\sup_{I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))] = H[\eta(X^*_{I-(\varphi\beta_i\gamma_i-c)}(\xi))]$$

   only if

$$\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(s) := N_{I-(\varphi\beta_i\gamma_i-c)}(s) - \int_0^s \frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma_{I-(\varphi\beta_i\gamma_i-c)}](\tau)}{\varsigma_{I-(\varphi\beta_i\gamma_i-c)}(\tau)}$$

   for $s \in [0,\xi]$ is a $(\mathbb{N}, m)$-martingale. In this regard,

$$\varsigma(s) := H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}\left[\frac{dm}{dQ_{I-(\varphi\beta_i\gamma_i-c)}}|_{\mathcal{N}_s}\right] = (H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)|\mathcal{N}_s])^{-1}$$

   for $s \in [0,\xi]$

2. However, Assuming an optimal portfolio $I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}$ exists, then

$$Z(s) = \int_0^s \omega(\tau)d^-D(\tau)$$

   is $(\mathbb{N}, m)$-semimartingale

3. Supposing an optimal $I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}$ exists such that $\omega \neq 0$ for

$$a.a(s,w) \in [0,\xi] \times \Omega$$

   then $D(\tau)$ is an $(\mathbb{N}, m)$-semimartingale

Proof

1. If $I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}_{\mathbb{N}}$ is optimal, then by Definition 4.6.3, $N_{I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, Q_{I-(\varphi\beta_i\gamma_i-c)})$-martingale with

$$F_{I-(\varphi\beta_i\gamma_i-c)}(\xi) = \frac{\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)}{H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)]}$$

   and (4.73) respectively, and by Girsanov, we have

$$\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau) := N_{I-(\varphi\beta_i\gamma_i-c)}(\tau) - \int_0^\tau \frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma](s)}{\varsigma(s)}$$

for $\tau \in [0,\xi]$ is a $(\mathbb{N}, m)$-martingale with

$$\varsigma(\tau) := H_{Q_{I-(\varphi\beta_i\gamma_i-c)}}[\frac{dm}{dQ_{I-(\varphi\beta_i\gamma_i-c)}}]|_{\mathcal{N}_\tau}$$

$$= H[(F_{I-(\varphi\beta_i\gamma_i-c)}(\xi))^{-1}\frac{F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)}{H[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)|_{\mathcal{N}_\tau}|_{\mathcal{N}_s}]}]$$

$$= (\vec{H}[F_{I-(\varphi\beta_i\gamma_i-c)}(\xi)|\mathcal{N}_\tau])^{-1}s \in [0,\xi]$$

conversely, if $N_{I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, Q_{I-(\varphi\beta_i\gamma_i-c)})$-martingale, then $N_{I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, m_{I-(\varphi\beta_i\gamma_i-c)})$-martingale and hence $I-(\varphi\beta_i\gamma_i-c)$ is optimal by Definition 4.6.3

2. is derived from (1)

3. By (2) recall that

$$Z(\tau) = \int_0^\tau \omega(s)d^-D(s)$$

is a semimartingale of $(\mathcal{N}, m)$. Then Assuming $\omega \neq 0$ for

$$a.a(s,D) \in [0,\xi] \times \hat{\Omega}$$

holds, we obtain

$$\int_0^s \omega^-(\tau)dZ(\tau) = \int_0^s \omega^-(\tau)\omega(\tau)d^-D(\tau) = \tilde{D}(\tau)$$

is a martingale of $(\mathcal{N}, m)$ also.

Theorem 4.6.1 indicate a clear connection between the optimal
$I-(\varphi\beta_i\gamma_i-c)$ and the decomposition of the semimartingale D with respect to $\mathbb{N}$. We prove this in the context of portfolio diversification.

Theorem 4.6.2

1. Given that $I-(\varphi\beta_i\gamma_i-c)$ is optimal for

$$\sup_{I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_\mathbb{N}} H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))] = H[\eta(X^*_{I-(\varphi\beta_i\gamma_i-c)}(\xi))]$$

Then D is a semimartingale with respect to $\mathbb{N}$ with a decomposition

$$dD(\tau) = d\hat{D}(\tau)+[\omega(\tau)I-(\varphi\beta_i\gamma_i-c)(\tau)-\frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}]d\tau+\frac{[N_{I-(\varphi\beta_i\gamma_i-c)},\varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

where $\hat{D}$ is a $(\mathbb{N}, m)$-Brownian motion

2. in reverse, assume D is a semimartingale with respect to the filtration and probability measure $(\mathbb{N}, m)$ with a decomposition $dD(\tau) = d\hat{D}(\tau) + dA(\tau)$ and $\mathbb{N}$-adapted finite variation process A, for some $\mathbb{N}$-adapted process α. Assuming $\alpha(\tau)d\tau = dA(\tau) =$ that is, $dA(\tau)$ is absolutely continuous with respect to $d\tau$ then the solution $I - (\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ for

$$\omega(\tau)I - (\varphi\beta_i\gamma_i - c) + \frac{1}{\tilde{\omega}(\tau)\varsigma(\tau)} \frac{d\left[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma\right](\tau)}{d\tau}$$

$$= \alpha(\tau) + \frac{\vartheta(\tau) - \breve{\lambda}(\tau)}{\omega(\tau)}.$$

Then, $I - (\varphi\beta_i\gamma_i - c)$ is optimal for

$$\sup_{I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_{\mathcal{N}}} H[\eta(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))] = H[\eta(X^*_{I-(\varphi\beta_i\gamma_i-c)}(\xi))]$$

since the variation of the quadratic $\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}$ is absolutely continuous.

$$[\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}, \hat{N}_{I-(\varphi\beta_i\gamma_i-c)}](\tau) = \int_0^\tau \omega^2(s)ds$$

from

$$\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau) := N_{I-(\varphi\beta_i\gamma_i-c)}(\tau) - \int_0^\tau \frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma](s)}{\varsigma(s)}, \ \tau \in [0, \xi]$$

this implies $d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)$ is absolutely continuous with respect to $d\tau$. As such,

$$\frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\tilde{\omega}(\tau)\varsigma(\tau)} = \frac{1}{\tilde{\omega}(\tau)\varsigma(\tau)} \frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma]\tau d\tau}{d\tau}$$

Proof

Assuming $I - (\varphi\beta_i\gamma_i - c)$ is optimal, by Theorem 4.6.2, the equation below

$$\begin{aligned} \omega^{-1}(\tau)d\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau) &= dD(\tau) \\ +\omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)I - (\varphi\beta_i\gamma_i - c)(\tau))d\tau \\ -\frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\varsigma(\tau)}] \end{aligned} \tag{4.79}$$

$\tau \in [0, \xi]$ is $(\mathbb{N}, m)$-martingale. as much as the process of the quadratic variation $\omega^{-1}(\tau)d\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau)$ $\tau \in [0, \xi]$, is τ, $\tau \in [0, \xi]$ it implies that

$$d\hat{D}(\tau) := \omega^{-1}(\tau)d\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau), \ \ \tau \in [0, \xi]$$

is $\mathbb{N}, m$-Brownian motion and

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)I - (\varphi\beta_i(\tau)\gamma_i(\tau) - c) - \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}]d\tau + \frac{[N_{I-(\varphi\beta_i\gamma_i-c)}\varsigma](\tau)}{\omega\tau\varsigma(\tau)}$$

follows

1. Supposing the decomposition of D is $(\mathbb{N}, m)$-semimartingale

$$dD(\tau) = d\hat{D}(\tau) + dA(\tau)$$

with reference to (2). Set ν to be

$$\nu = \alpha(t) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}.$$

Then,

$$\begin{aligned} \omega^{-1}(\tau)d\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau) = dD(\tau) \\ +\omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)I - (\varphi\beta_i(\tau)\gamma_i - c)(\tau))d\tau \\ -\frac{d[N_{I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\varsigma(\tau)}] \end{aligned} \tag{4.80}$$

$\tau \in [0, \xi]$

$$dD(\tau) - dA(\tau) = d\hat{D}(\tau)$$

therefore

$$\omega^{-1}(\tau)d\hat{N}_{I-(\varphi\beta_i\gamma_i-c)}(\tau), \ \tau \in [0, \xi],$$

is $(\mathbb{N}, m)$-martingale in that manner, $I - (\varphi\beta_i\gamma_i - c)$
is optimal via Theorem 4.6.2

1. Given $\eta(x) = \frac{1}{h}x^h, \ \ x > 0$ where $h \in (0, 1)$ we have

$$\eta'(X_{I-(\varphi\beta_i\gamma_i-c)+h\varpi}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)+h\varpi}(\xi)|M(h)| = X_{I-(\varphi\beta_i\gamma_i-c)^h+h\varpi}(\xi)|M(h)$$

and condition (4) in our earlier Definition is satisfied if

$$\sup_{h\in(-\delta,\delta)} H[(X^h_{I-(\varphi\beta_i\gamma_i-c)+h\varpi}(\xi)|M(h)|)^{\hat{p}}] < \infty$$

for $\hat{p} > 1$ then set

$$X_{I-(\varphi\beta_i\gamma_i-c)+h\varpi}(\xi) = X_{I-(\varphi\beta_i\gamma_i-c)}(\xi)N(h).$$

From the Holders inequality,

$$H[(X^h_{I-(\varphi\beta_i\gamma_i-c)+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))^{h\hat{p}\tilde{a}_1\tilde{b}_1}])^{\frac{1}{\tilde{a}_1\tilde{b}_1}}$$

$$(H[(N(h))^{h\hat{p}\tilde{a}_1\tilde{b}_2}])^{\frac{1}{\tilde{a}_1\tilde{b}_2}}(H[(|M(h)|)^{\hat{p}\tilde{a}_2}])^{\frac{1}{\tilde{a}_2}}$$

where $\tilde{a}_1, \tilde{a}_2 : \frac{1}{\tilde{a}_1} + \frac{1}{\tilde{a}_2} = 1$ and $\tilde{b}_1, \tilde{b}_2 : \frac{1}{\tilde{b}_1} + \frac{1}{\tilde{b}_2} = 1$ Choosing $\tilde{a}_1 = \frac{2}{2-\hat{p}}$ $\tilde{a}_2 = \frac{2}{\hat{p}}$ and also $\tilde{b}_1 = \frac{2-\hat{p}}{h\hat{p}}$, $\tilde{b}_2 = \frac{2-\hat{p}}{2-\hat{p}-h\hat{p}}$ for some $\hat{p} \in (1, \frac{2}{h+1})$. Hence

$$H[(X^h_{I-(\varphi\beta_i\gamma_i-c)+h\varpi}(\xi)|M(h)|)^{\hat{p}}] \leq (H[(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))^2])^{\frac{h\hat{p}}{2}}$$

$$(H[(N(h))^{\frac{2h\hat{p}}{2-\hat{p}-h\hat{p}}}])^{\frac{2-\hat{p}-h\hat{p}}{2}}(H[(|M(h)|^2)])^{\frac{\hat{p}}{2}}$$

supposing $N_{I-(\varphi\beta_i\gamma_i-c)}(\xi)$ in

$$N(t) = x\exp\{\int_0^s [\lambda(\tau) + (\varsigma(\tau) - \lambda(\tau))\beta_i(\tau)\gamma_i(\tau)$$
$$-\frac{1}{2}\omega^2(\tau)(I - (\varphi\beta_i(\tau)\gamma_i - c(\tau))^2]d\tau + \int_0^s \omega(\tau)I - (\varphi\beta_i(\tau)\gamma_i(\tau) - c)d^- D(\tau) \tag{4.81}$$

satisfies

$$H[(N_{I-(\varphi\beta_i\gamma_i-c)}(\xi))^2] < \infty$$

Then (4) and (5) of our earlier definitions is valid if

$$\sup_{h\in-\delta,\delta} H[(N(h))\frac{2h\hat{p}}{2-\hat{p}-x\hat{p}}] < \infty$$

however

$$\sup_{h\in-\delta,\delta} H[(N(h))\frac{2h\hat{p}}{2-\hat{p}-x\hat{p}}] < \infty$$

holds if for example

$$H\exp\{k\int_0^s [|\vartheta(\tau) - \lambda(\tau)| + |I - (\varphi\beta_i\gamma_i - c)(\tau)|d\tau\}] < \infty \ \forall \ k > 0$$

however,

$$H[(N_{I-(\varphi\beta_i\gamma_i-c)}(\xi))^2] < \infty$$

is equally verify for $k > 0$

$$\begin{aligned} &H\exp\{k(\int_0^\xi [|\vartheta(\tau) - \lambda(\tau)| + |I - (\varphi\beta_i\gamma_i - c)(\tau)|d\tau])\} \\ &\quad +|\int_0^\xi I - (\varphi\beta_i\gamma_i - c)(\tau)\omega(\tau)dD(\tau)| < \infty \end{aligned} \tag{4.82}$$

4.7 Large scale investor with budget constraints and consumption under insurance cover

In this section, we discuss the large scale investors with a similar budget constraints such as allocation of funds and consumption under which some financial aids from huge financial institutions are provided as insurance. The essence of this concept is to encourage large investors such as property developers. As a result, a sensitive investor splits his investment network on different assets to avoid less return as expected from assets. we defined

$$H : \mathbb{R}^n \to \langle \beta, \gamma \rangle, \subseteq \mathbb{R}$$

where

$$\langle \hat{\beta, \hat{\gamma}} \rangle = \langle (\beta_1 \beta_2), (\gamma_1 \gamma_2), \cdots \rangle = \sum_{i=1}^{n} I - (\varphi \beta_i \gamma_i - c)$$

where $\sum_{i=1}^{n} \nu_i I - (\varphi \beta_i \gamma_i - c)$ and ν_i is a confine placed before investors on the size of assets to be held due to costs of transactions. $\nu_i = 1$ if assets i is chosen in the portfolio, and 0 otherwise. We denote φ as the constraint subject to investors budget and c is the consumption of the investor, while I represents insurance cover Øksendal (2006), Mark Schroder (1999) and Samuelson (1969). Thus, we defined $\mathcal{A}_{\mathbb{N}}$ as the set of admissible portfolio for the investor and explicitly expressed in the definition below:

Definition 4.7.1

$\mathcal{A}_{\mathbb{N}}$ is the set of admissible portfolios expressed as

1. all $\sum_{i=1}^{n} I - (\varphi \beta_i \gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ is c*á*gl*á*d and $\mathbb{N}$-adapted

2. $\sum_{i=1}^{n} I - (\varphi \beta_i \gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$.

$$H \int_0^{\xi} |\vartheta(\tau) - \lambda(\tau)| | (I - (\varphi \beta_i(\tau) \gamma_i(\tau) - c) | +$$

$$\sum_{i=1}^{n} I - (v \varphi \beta_i(\tau) \gamma_i(\tau) - c)^2 \, d\tau < \infty$$

3. $I - (\varphi \beta_i \gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$,the product $\sum_{i=1}^{n} I - (\varphi \beta_i \gamma_i - c) \, \omega$ is forward integrable and c*á*gl*á*d

4. $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_{\mathbb{N}}$, then

$$0<H[\eta'(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)]<\infty$$

where $\eta'(l)=\frac{d}{dl}\eta(l)$.

5. for all $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c),\varpi\in\mathcal{A}_{\mathbb{N}}$, there exist $\delta>0$, with a bounded ϖ, such that $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi\in\mathcal{A}_{\mathbb{N}}$ in all $j\in(-\delta,\delta)$ as such, the entire family

$$\eta'(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi)|N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi)|_{j\in(-\delta,\delta)}$$

is uniformly integrable, where

$$N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau):=\int_0^s[\lambda(\tau)-\lambda(\tau)-\omega^2(\tau)\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)(\tau)]d\tau$$
$$+\int_0^s\omega(s)dD(\tau),$$

6. ϖ portfolio permits buy-hold sell strategy, of

$$\varpi(s)=\alpha I(\tau,\tau+f](t),\quad t\in[0,\xi]$$

with $0\le\tau<s+f\le\xi$ and $\mathcal{N}_\tau$-measurable process α belonging to $\mathcal{A}_{\mathbb{N}}$. Then $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_{\mathbb{N}}$ is optimal if

$$H[\eta(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))]=H[\eta(L_{\nu_1}(\xi))]$$

for a bounded $\varpi\in\mathcal{A}_{\mathbb{N}}$ and $y\in(-\delta,\delta)$ with $\delta>0$ given in (5).

Definition 4.7.2

Assuming ϖ is a forward integrable stochastic process and N a random variable, then the product $N\varpi$ is forward integeable stochastic process. Thus,

$$\int_0^\xi N\varpi(\tau)d^-D(\tau)=N\int_0^\xi\varpi(\tau)d^-D(\tau)$$

where $\varpi=X(\tau)\nu(\tau)^*$ in this sense, $\nu_1^*=\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)$. Firstly, $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)$ is optimal. For a bounded $\varpi\in\mathcal{A}_{\mathbb{N}}$ we have

$$0=\frac{d}{dj}H[\eta(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))]|_{j=0}$$

$$
\begin{aligned}
0 = H[\eta'(X(\xi))X_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi) \\
\int_0^\xi \varpi(\tau)[\vartheta-\lambda(\tau)-\omega^2(s)\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)(\tau)]d\tau \\
+\int_0^\xi \varpi(\tau)\omega(\tau)d^-D(\tau)]
\end{aligned}
\tag{4.83}
$$

fixing $\tau, f : 0 \leq \tau < \tau+\tau \leq \xi$ and choosing $\varpi(s) = \alpha I(\tau,\tau+f](s),\ t\in[0,\xi]$, for any stochastic amount α of $\mathcal{N}_\tau$ bounded. Our equation (4.83) becomes

$$
\begin{aligned}
0 = H[\eta'(X_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi) \\
\int_\tau^{\tau+f}[\vartheta-\lambda(s)-\omega^2(s)\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)(s)]ds \\
+\int_\tau^{\tau+f}\omega(s)d^-D(s)\alpha]
\end{aligned}
\tag{4.84}
$$

this holds in all α, then

$$
H[F_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)(N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau+f)-N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau))|_{\mathcal{N}_\tau}]=0
$$

where

$$
F_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)=\frac{\eta'(X_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)}{H[\eta'(X_{\sum_{i=1}^n I-(-c)\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)]}
$$

and

$$
\begin{aligned}
N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau) := \exp\int_0^\tau[\vartheta-\lambda(s)-\omega^2(s)\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)(s)]ds \\
+\int_0^\tau\omega(s)d^-\hat{D}(s)]
\end{aligned}
\tag{4.85}
$$

that is,

$$
\begin{aligned}
0 = H[F_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)(\int_s^{s+f}\alpha(\tau)-\lambda(\tau)-\tilde{\omega}^2(\tau)\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)(\tau)\}d\tau \\
+\int_s^{s+f}\omega(\tau)d^-D(\tau))|_{\mathcal{N}_\tau}]
\end{aligned}
\tag{4.86}
$$

$$
\begin{aligned}
H_{Q_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}}[N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau+f) \\
-N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau)|_{\mathcal{N}_\tau}] \\
0 = H[F_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)|_{\mathcal{N}_\tau}]^- H[F_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\xi)(N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau+f) \\
-N_{\sum_{i=1}^n I-(\varphi\beta_i\gamma_i-c)}(\tau))|_{\mathcal{N}_\tau}]
\end{aligned}
\tag{4.87}
$$

since $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)$ is $\mathcal{N}_\tau$-adapted, this gives

$$H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau+f)|_{\mathcal{N}_\tau}] = N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau).$$

Hence, $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)$ is an $(\mathcal{N}_\tau, Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)})$-martingale.
Let the probability measure $Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ on $\mathcal{N}_\tau$ be

$$dQ_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)} = F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)dm$$

and $H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}(\tau)$ an expectation of $Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ So,

$$H[F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)(N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau+f) - N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau))|_{\mathcal{N}_\tau}] = 0$$

written as

$$H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau+f) - N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)|_{\mathcal{N}_\tau}] = 0$$

$N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)$, $\tau \in [0,\xi]$, is a $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)})$-martingale, that is, $\mathbb{N}$-filtration under the probability measure $Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ measure it can equally be stated as follows that suppose $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)})$-martingale. Then,

$$H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau+f) - N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)_{\mathcal{N}_\tau}] = 0$$

in all τ, f and then $0 \leq \tau < \tau + f \leq \xi$. Similarly,

$$H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau+f) - N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)\alpha] = 0$$

for $\mathcal{N}_\tau$-measurable process α. therefore

$$\begin{aligned} 0 = H[\eta'(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi) \\ \int_\tau^{\tau+f} [\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(s)]ds \\ + \int_\tau^{\tau+f} \omega(s)d^- D(s)\alpha] \end{aligned} \tag{4.88}$$

holds. taking linear combination

$$\begin{aligned} 0 = H[\eta'(X_{I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{I-(\varphi\beta_i\gamma_i-c)}(\xi) \\ \int_0^{\xi} \varpi(s)[\vartheta - \lambda(s) - \omega^2(s)\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(s)]ds \\ + \int_0^{\xi} \varpi(s)\omega(s)d^- D(s)] \end{aligned} \tag{4.89}$$

remain valid for all cáglád step processes $\varpi \in \mathcal{A}_{\mathbb{N}}$ from assumption (1) and (5) we have (4.89) still holds for a bounded $\varpi \in \mathcal{A}_{\mathbb{N}}$. Provided the function

$$y \to H[\eta(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))],\ j \in (-\delta, \delta)$$

its maximum is attained at $j = 0$. Thus,

$$0 = \frac{d}{dj} H[U(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)+j\varpi}(\xi))]|_{j=0}$$

Definition 4.7.3

A stochastic process $\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ is optimal as far as the stochastic process

$$\begin{aligned} N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) := \exp \int_0^{\tau} [\vartheta - \lambda(s) - \omega^2(s) \sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(s)]ds \\ + \int_0^{\tau} \omega(s) d^- D(s)],\ \tau \in [0, \xi] \end{aligned} \tag{4.90}$$

is $\mathbb{N}, Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$-martingale. By the application of Girsanov's theorem, it is equally stated as follows

Theorem 4.7.1

1. The process $\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ for

$$\sup_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}_{\mathbb{N}}} H[\eta(L_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))] = H[\eta(L^*_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))]$$

is optimal if

$$\begin{aligned} \hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) := N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) \\ - \int_0^{\tau} \frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \upsilon_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}](s)}{\upsilon_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(s)},\ \tau \in [0, \xi] \end{aligned} \tag{4.91}$$

is a $(\mathbb{N}, m)$-martingale. in this sense,

$$\begin{aligned} \upsilon_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) := H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}} \left[\frac{dm}{dQ_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}} |_{\mathcal{N}_\tau} \right] \\ = (H[F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)|\mathcal{N}_\tau])^{-1} 1\ t \in [0, \xi] \end{aligned} \tag{4.92}$$

2. However, If an optimal portfolio $\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) \in \mathcal{A}$ exists, then the process

$$\varsigma(s) = \int_0^{\tau} \omega(s) d^- D(s)$$

is a $(\mathbb{N}, m)$-semimartingale

3. supposed $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i - c) \in \mathcal{A}$ of optimal value exists and $\omega \neq 0$ for

$$a.a(s,w) \in [0,\xi] \times \grave{\Omega}$$

then $D(\tau)$ is an $(\mathbb{N}, m)$-semimartingale

Proof

1. If $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i - c) \in \mathcal{A}_{\mathbb{N}}$ is optimal, by Definition 4.7.3, then $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)})$-martingale with
$$F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi) = \frac{\eta'(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)}{H[\eta'(X_{\sum_{i=1}^{n} I-(-c)\varphi\beta_i\gamma_i}(\xi))X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)]}$$
and (4.85). By applying theorem of Girsanov,
$$\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) := N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) - \int_0^\tau \frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](s)}{\varsigma(s)}$$
is $(\mathbb{N}, m)$-martingale
$$\begin{aligned} \varsigma(\tau) &:= H_{Q_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}[\frac{dm}{dQ_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}}]|_{\mathcal{N}_\tau} \\ = H[(F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))^{-1} &\frac{F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)}{H[F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)|_{\mathcal{N}_\tau}|_{\mathcal{N}_\tau}]}] \\ &= (H[F_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)|\mathcal{N}_t])^{-1}, \tau \in [0,\xi] \end{aligned} \tag{4.93}$$
conversely, if $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, Q_{I-(\varphi\beta_i\gamma_i-c)})$-martingale, then $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ is a $(\mathbb{N}, m_{I-(\varphi\beta_i\gamma_i-c)})$-martingale and hence $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i - c)$ is optimal by Definition 4.7.3

2. is derived from (1)

3. By (2) it is clear that
$$Y(t) = \int_0^t \omega(\tau)d^- D(\tau)$$
is a $(\mathbb{N}, m)$-semimartingale. supposing $\omega \neq 0$ for
$$a.a(s,w) \in [0,\xi] \times \Omega$$
its valid, then
$$\int_0^s \omega^-(\tau)dY(\tau) = \int_0^s \omega^-(\tau)\omega(\tau)d^- D(\tau) = D(\tau)$$
is $(\mathbb{N}, m)$-semimartingale also.

Theorem 4.7.1 furnishes a clear connection between $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)$ optimal portfolio and decomposition of semimartingale D with respect to $\mathbb{N}$. we prove this in the context of portfolio diversification.

Theorem 4.7.2

1. Given that $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)$ is optimal, then the decomposition of semimartingale D with respect to $\mathbb{N}$ is

$$dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)(\tau) - \frac{\vartheta(\tau)-\lambda(\tau)}{\tilde{\omega}(\tau)}]d\tau$$

$$+\frac{[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)},\varsigma](\tau)}{\tilde{\omega}\tau\varsigma(\tau)}$$

   where $\hat{D}$ is a $(\mathbb{N}, m)$-Brownian motion

2. in reverse, assume the semimartingale D with respect to $(\mathbb{N}, m)$ decomposes as $dD(\tau) = d\hat{D}(\tau) + dA(\tau)$, where $\hat{D}$ is $(\mathbb{N}, m)$ Brownian motion and $\mathbb{N}$-adapted process A of finite variation. Assume $dA(\tau) = \alpha(\tau)d(\tau)$ for some $\mathbb{N}$-adapted process α that is, $dA(\tau)$ is absolutely continuous with respect to $d(\tau)$ then there is a solution $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c) \in \mathcal{A}_{\mathbb{N}}$ of the form

$$\omega(\tau)\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c) + \frac{1}{\omega(\tau)\varsigma(\tau)}\frac{d\left[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)},\varsigma\right](\tau)}{d\tau}$$

$$= \alpha(\tau) + \frac{\vartheta(\tau)-\lambda(\tau)}{\omega(\tau)}$$

in this sense $\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)$ proves optimal for

$$\sup_{I-(\varphi\beta_i\gamma_i-c)\in\mathcal{A}_{\mathbb{N}}} H[\eta(X_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))] = H[\eta(X^*_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))]$$

since the quadratic variation of $\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}$ is absolutely continuous that is,

$$[\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)},\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}](\tau) = \int_0^\tau \omega^2(s)ds$$

from

$$\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) := N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) - \int_0^\tau \frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)},\varsigma](s)}{\varsigma(s)}$$

for $\tau \in [0, \xi]$ then $d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)$ is absolutely continuous in connection with $d\tau$.

$$\frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\vec{\omega}(\tau)\varsigma(\tau)} = \frac{1}{\vec{\omega}(\tau)\varsigma(\tau)} \frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)d\tau}{d\tau}$$

Proof

Assuming $\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)$ is optimal, by Theorem 4.7.2

$$\begin{aligned} \omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) = dD(\tau) \\ +\omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(\tau))d\tau \\ -\frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\varsigma(\tau)}] \end{aligned} \tag{4.94}$$

$\tau \in [0, \xi]$ is $(\mathbb{N}, m)$-martingale and $\omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau)$ $\tau \in [0, \xi]$, is τ, $\tau \in [0, \xi]$ a quadratic variation then

$$d\hat{D}(\tau) := \omega^{-1}(\tau)d\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau), \quad \tau \in [0, \xi]$$

is $(\mathbb{N}, m)$-Brownian motion and

$$\begin{aligned} dD(\tau) = d\hat{D}(\tau) + [\omega(\tau)\sum_{i=1}^{n} I - (\varphi\beta_i(\tau)\gamma_i(\tau) - c) - \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}]d\tau \\ +\frac{[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\omega\tau\varsigma(\tau)} \end{aligned}$$

holds

1. Assume D is an $(\mathbb{N}, m)$-semimartingale with a decomposition

   $$dD(\tau) = d\hat{D}(\tau) + dA(\tau)$$

   with referencing to (2). Set ν

   $$\nu = \alpha(\tau) + \frac{\vartheta(\tau) - \lambda(\tau)}{\omega(\tau)}$$

   Then

   $$\begin{aligned} \omega^{-1}(t)d\hat{N}_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\tau) = dD(\tau) \\ +\omega^{-1}(\tau)[(\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau)\sum_{i=1}^{n} I - (\varphi\beta_i(\tau)\gamma_i - c)(\tau))d\tau \\ -\frac{d[N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}, \varsigma](\tau)}{\varsigma(\tau)}] \end{aligned} \tag{4.95}$$

$\tau \in [0, \xi]$

$$dD(\tau) - dA(\tau) = d\hat{D}(\tau)$$

consequently

$$\omega^{-1}(\tau) d\hat{N}_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)}(\tau), \quad \tau \in [0, \xi],$$

is a $(\mathbb{N}, m)$-martingale then $\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)$ is optimal from Theorem 4.7.2

1. Given $\eta(x) = \frac{1}{h} x^h, \quad x > 0$

where $h \in (0, 1)$ we have

$$\begin{aligned} \eta'(X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) + h\varpi}(\xi)) X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) + h\varpi}(\xi) |M(h)| \\ = X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)^h + h\varpi}(\xi) |M(h) \end{aligned} \tag{4.96}$$

and condition (4) in our earlier Definition is satisfied if

$$\sup_{h \in (-\delta, \delta)} H[(X^h_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) + h\varpi}(\xi) |M(h)|)^p] < \infty$$

for $p > 1$. We set

$$X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) + h\varpi}(\xi) = X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)}(\xi) N(h)$$

where

$$N_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)}(h) := \exp \int_0^{\xi} [\vartheta(\tau) - \lambda(\tau) - \omega^2(\tau) \sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(\tau)] d\tau$$

$$+ \int_0^t \omega) dD(\tau).$$

From the Holders inequality, we have

$$H[(X^h_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) + h\varpi}(\xi) |M(h)|)^{\hat{p}}] \leq (H[(X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)}(\xi))^{h\hat{p}a_1b_1}])^{\frac{1}{a_1b_1}}$$

$$(H[(N(h))^{h\hat{p}a_1b_2}])^{\frac{1}{a_1b_2}} (H[(|M(h)|)^{\hat{p}a_2}])^{\frac{1}{a_2}}$$

where $a_1, a_2 : \frac{1}{a_1} + \frac{1}{a_2} = 1$ and $b_1, b_2 : \frac{1}{b_1} + \frac{1}{b_2} = 1$. Then we can choose $a_1 = \frac{2}{2-p}$ $a_2 = \frac{2}{\hat{p}}$ and also $b_1 = \frac{2-\hat{p}}{h\hat{p}}$, $b_2 = \frac{2-\hat{p}}{2-\hat{p}-h\hat{p}}$ for some $\hat{p} \in (1, \frac{2}{h+1})$. Hence,

$$H[(X^h_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c) + h\varpi}(\xi) |M(h)|)^{\hat{p}}] \leq (H[(X_{\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)}(\xi))^2])^{\frac{hp}{2}}$$

$$(H[(N(h))^{\frac{2h\hat{p}}{2-\hat{p}-h\hat{p}}}])^{\frac{2-\hat{p}-h\hat{p}}{2}} (H[(|M(h)|^2)]))^{\frac{\hat{p}}{2}}$$

if $N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi)$ in

$$N(s) = x\exp\{\int_0^s \lambda(\tau) + (\vartheta(\tau) - \lambda(\tau))\sum_{i=1}^{n} I - (\varphi\beta_i(\tau)\gamma_i - c)(\tau)\}$$

$$-\frac{1}{2}\omega^2(\tau)(\sum_{i=1}^{n} I - (\varphi\beta_i(\tau)\gamma_i(\tau) - c))^2 d\tau$$
$$+\int_0^s \omega(\tau)\sum_{i=1}^{n} I - (\varphi\beta_i(\tau)\gamma_i(\tau) - c)d^- D(\tau) \tag{4.97}$$

satisfies

$$H[(N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))^2] < \infty.$$

Then, the condition (4) and (5) of our earlier definitions holds if

$$\sup_{h\in -\delta,\delta} H[(N(h))\frac{2hp}{2-p-xp}] < \infty$$

however

$$\sup_{h\in -\delta,\delta} H[(N(h))\frac{2hp}{2-p-xp}] < \infty$$

holds if for example

$$H\exp\{k\int_0^s [|\vartheta(\tau) - \lambda)| + |\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(\tau)|d\tau\}] < \infty \;\forall\; k > 0$$

however,

$$H[(N_{\sum_{i=1}^{n} I-(\varphi\beta_i\gamma_i-c)}(\xi))^2] < \infty$$

is equally verify for all $k > 0$

$$H\exp\{k(\int_0^\xi [|\vartheta(\tau) - \lambda(\tau)| + |\sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(\tau)|d\tau)\}$$
$$+|\int_0^\xi \sum_{i=1}^{n} I - (\varphi\beta_i\gamma_i - c)(\tau)\omega(\tau)dD(\tau)| < \infty \tag{4.98}$$

Example 4.7.1

If the expected value of the investors utility is given as

$$\frac{\hat{\zeta}}{1+\alpha} - \frac{1-\hat{\zeta}}{1-\alpha} = 0$$

then

$$\hat{\zeta}(1-\alpha) = (1-\hat{\zeta})(1+\alpha)$$

$$\hat{\zeta} - \alpha\hat{\zeta} = 1 + \alpha - \hat{\zeta} - \alpha\hat{\zeta}$$

$$\hat{\zeta} - \alpha\hat{\zeta} + \alpha\hat{\zeta} = 1 + \alpha - \hat{\zeta}$$

$$\hat{\zeta} + \hat{\zeta} - \alpha\hat{\zeta} + \hat{\alpha}\hat{\zeta} = 1 + \alpha$$

$$2\hat{\zeta} = 1 + \alpha$$

$$\alpha = (2\hat{\zeta} - 1)$$

when $\hat{\zeta} = 0.5$ $\alpha = 0$. Thus, if $\hat{\zeta} > \frac{1}{2}$ the investor will distribute $100(2\hat{\zeta} - 1)\%$ of his capital. if $0 \leq \hat{\zeta} \leq \frac{1}{2}$,then $E[U(X)]$ is the greatest value when $\alpha = 0$, i.e. when no investment is made by the investor. But, when $\hat{\zeta} = 0.6 \quad \alpha = 0.2$, similarly when $\hat{\zeta} = 0.7 \quad \alpha = 0.4$, when $\hat{\zeta} = 0.8 \quad \alpha = 0.6$ when $\hat{\zeta} = 0.9 \quad \alpha = 0.8$

4.8 Concept of equivalent martingale measure

In this section, equivalent market measure (EMM) and its relation to the non-existence of arbitrage were established. It is shown that any EMM is equal to the risk of market price and denotes the price of risk-neutral on the derivatives Delbaen (2006). Regularities conditions which assures the reality of market price were imposed on the drift parameter (μ) volatility (σ), and the interest (r)rate . However, if the amount of securities have similar number of underlying stochastic process that is, $\tilde{N} = \hat{d} \leq \infty$ then it is a complete market and establish a unique measure of martingale. (Monteiro *et al.* (2008)).

The underlying stochastic is Brownian. Considering a financial market with deterministic non risky asset $B_t = e^{rt}$, $r \geq 0$, and $\hat{d} \geq 1$ following $S_t^i = S_0^i e^{W_t^i}$. The non-risk account is used for reduction. The set of information is $(\mathcal{F}_t)_t \in [0, T]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of probability with filtration $\mathbb{F} := \{\mathcal{F}_t\}$ $t \geq 0$ generated by random variables S_s, for $s \leq t$ for all p-zero.

Considering the risky assets price $S_t = S_0 e_t^W, t \in [0, T]$ with

$$W_t = \int_0^t (\mu_s - \frac{\sigma^2}{2})ds + \int_0^t S_t^2 \sigma_s dW_t \; t \in [0, T], \tag{4.99}$$

where, $dW_t \quad t \in [0, T]$, is a Brownian motion. The average return rate μ_t and σ_t are adapted and measurable obeying the integrability conditions

$$\int_0^t |\mu_t| dt < \infty, \quad \int_0^t \sigma_t^2 dt < \infty$$

a.s. By Itô, we have that S_t satisfies:

$$ds_t = \mu_t s_t dt + \sigma_t s_t dW_t \quad t \in [0, T], \tag{4.100}$$

The bond $\beta_t, t \in [0, T]$ price, moves according to $d\beta_t = r_t \beta_t dt, \ \ \beta_0 = 1$, where r_t is positive satisfying the integrability condition $\int_0^t r_t dt < \infty$,a.s that is,

$$\beta_t = \exp \int_0^t r_s ds. \tag{4.101}$$

An investor commence its business with $x \geq 0$ of the description above. Let α_t represents the size of riskless assets and β_t on stocks owned by the investor at t. The couple $\phi_t = (\alpha_t \ \beta_t), \quad t \in [0, T]$ is called a portfolio or trading strategy, and we assume that α_t and β_t are adapted and measurable then

$$\int_0^t |\beta_t \mu_t| dt < \infty \quad \int_0^t \beta_t^2 S_t^2 \sigma_t^2 dt < \infty \quad \int_0^t |\alpha_t| r_t dt < \infty \quad a.s. \tag{4.102}$$

Then $x = \alpha_0 + \beta_0 S_0$ the portfolio value) is

$$V_t(\phi) = \alpha_t \beta_t + \beta_t S_t \tag{4.103}$$

The gain $G_t(\phi)$ the investor realises through the portfolio ϕ up to t is

$$G_t(\phi) = \int_0^t \alpha_s d\beta_s + \int_0^t \beta_s dS_s \tag{4.104}$$

Then the portfolio ϕ is self-funding when there is no fresh investment nor withdrawal. Automatically the value equals initial investment in addition to the gain:

$$V_t^\phi = x + \int_0^t \alpha_s d\beta_s + \int_0^t \beta_s dS_s \tag{4.105}$$

We add some criteria on the portfolio to prevent arbitrage opportunities.

Definition 4.8.1 P and γ are measures of two probability on $(\Omega, \mathcal{F})$ Then:

i γ and P are equivalent if γ is completely continuous for P vice versa with P and γ i.e. $\gamma \equiv P$ only if γ is utterly continuous in honor of P and vice versa.

Remark 4.8.1: if γ is entirely continuous with regards to P then by the Randon-Nikodym theorem, there is a unique $Z \in L^1(\Omega, \mathcal{F}, P)$ which is nonnegative almost everywhere and

$$\gamma(A) = \int_A Z(W) dp(W) \ \forall \ A \in \mathcal{F}$$

Function Z is Randon-Nikodym derivative of γ in relation with P and is often denoted by $\frac{d\gamma}{dP} = Z(W)$. is a condition.

Definition 4.8.2

If υ and ϱ are equal then, $\upsilon(A) = 0$ whenever $\varrho(A) = 0$ hence, there is a strict positive function $Z \in L^1(\Omega, \mathcal{F}, \varrho)$ such that

$$\upsilon(A) = \int_A Z(W) d\varrho(W) \ \ \forall \ \in \mathcal{F}$$

and

$$\varrho(A) = \int_A \frac{1}{Z}(W)\upsilon d(W) \ \ \forall \ A \in \mathcal{F}$$

υ on $(\Omega, \mathcal{F})$ takes equal measures of martingale if:

1. υ is identical with ϱ
2. The price reduction $\tilde{S}(t) = e^{-rt}s(t) \ t \in [0, T]$ is a υ-martingale.

Is a condition.

Notation: Let all equal martingale measure set be $f(P)$.

Definition 4.8.3 (Existence of EMM):

We prove that arbitrage opportunities are prevented in the market if there is an identical measures of martingale $\gamma \in f(P) \neq \phi$,
for which the assets price S_t is a martingale.

Theorem 4.8.1(change of measures)

A probability measure γ is equal with P if asset price $S \in {l^2}_{loc}$ exists for

$$\frac{d\gamma}{dP} | \mathcal{F}_t^W = Z_t$$

and

$$dz_t = -Z_t S_t dW_t \ \ t \in [0, T]$$

where W^S is defined as

$$dW_t^s = dW_t - \varrho S_s dt$$

Is a $(\Omega, \mathcal{F}, \gamma, \mathcal{F}_t^W)$ Brownian

Proof:

the existence of a d-dimensional process of the martingale representation theorem $\tilde{S} \in {l^2}_{loc} \ \mathcal{F}_t^W$ such that

$$\frac{d\gamma}{dP} | \mathcal{F}_t^W = Z_t$$

and

$$\begin{aligned} dz_t &= -Z_t\tilde{S}_t d\tilde{W}_t \\ &= -Z_t\tilde{S}_t(A^{-1})dW_t \\ &= -Z_t S_t dW_t \end{aligned} \tag{4.106}$$

where

$$S_t = (A^{-1})^*\tilde{S}_t$$

.

$$Z_t = \exp\left(-\int_0^t S_s dW_s - \frac{1}{2}\int_0^t < \varrho^{S_s,S_s} > ds\right) \tag{4.107}$$

where $< \tilde{S}_t, A^{-1}dW_t >=< (\vec{A}^{-1})^*\tilde{S}_t dW_t >$. and $W_t = A\tilde{W}, \quad \varrho = A^*A\ \tilde{W}_t^{\tilde{S}}$ takes the form $d\tilde{W}_t^{(\tilde{S}_t)} = d\tilde{W}_t - \tilde{S}_t dt, \tilde{t} \in [0, \grave{T}]$.
Is Brownian $\tilde{S}_t = A^*S_t$, upon A multiplication, we have $dW_t^S := Ad\tilde{W}_t^{\tilde{s}} = dW_t - \varrho S_t dt$ is a γ-Brownian motion

Remark 4.8.2

We define a d-dimensional Brownian motion $W_t \quad t \in [0, T]$ on the probability space $(\hat{\Omega}, \tilde{\mathcal{F}}, \mathcal{P}, \mathcal{F}_t)$. Let $S \in l^2_{loc}$ be a process of d-dimensional and an exponential martingale associated with the asset price S be define as

$$Z_t^s = \exp\left(\int_0^t S_s dW_s - \frac{1}{2}\int_0^t |S_s|^2 ds\right)2 \ \ t \in [0, T] \tag{4.108}$$

And note that (".") represents a product of scalar in $\mathbb{R}^d$. by the It$\hat{o}$ formula we have

$$dZ_t^s = -Z_t^s S_t dW_t \tag{4.109}$$

then Z^S is martingale. Since Z^S is positive.

$$E_\gamma[Z_t^s] \le E[Z_0^s] = 1 \ \ t \in [0, T].$$

And $(Z_t^s)_t \in [0, T]$ is a strict martingale if $E_\gamma[Z_T^s] = 1$.
which implies the price $\tilde{S}(t)$ is martingale relative to γ i.e. $\gamma \in f(P)$. hence one of the similar measure $\gamma \in f(P)$ and a martingale exists, showing that the risky-assets driven by Brownian motion is arbitrage-free.

Lemma 4.8.1

Assume that Z_t^s in (4.108) is a p-martingale and γ is a measure of probability by $\frac{d\gamma}{dP} = Z_t^s$. Then the process $(M_t)_{T\geq 0}$ is a martingale of γ.

Proof

Since Z^S is strictly positive and adapted, then M is adapted and so is MZ^S. Since $Z^S \ \ isP$-martingale, and M is integrable of γ if MZ^S is p-integrable:

$$E^\gamma[|M_t|] = E^p[|M_t|Z_T^S] = E^p[E^p[|M_t Z_T^S|\mathcal{F}_t]]$$

$$= E^p[|M_t|E^p[Z_T^S|F_t]]$$

$E^\gamma[|M_t|Z_t^S]$ Similarly for $S \leq t$ we have

$$E^p[M_t Z_T^S|\mathcal{F}_S] = E^P[E^p|M_t Z_T^S|\mathcal{F}_s]$$

$E^p[|M_t Z_t^S|\mathcal{F}_s]$ Then by

$$E^\gamma[X|\mathcal{F}_t] = \frac{E^p[XZ_T^s|F_t]}{E^P[Z_T^s|\mathcal{F}_t]}$$

with $X = M_t$ we have

$$\begin{aligned} E^\gamma[M_t|\mathcal{F}_s] &= \frac{E^p[M_t Z_T^S|\mathcal{F}_s]}{E^P[Z_T^S|\mathcal{F}_s]} \\ &= \frac{E^P[M_t Z_T^s|\mathcal{F}_s]}{Z_s^s} \end{aligned} \tag{4.110}$$

Remark 4.8.3

From the above Lemma, The process

$$(Z_t^\gamma)^{-1} = \exp\left(\int_0^t S_s dW_s + \frac{1}{2}\int_0^t |S_s|^2 ds\right), \quad t \in [0, T] \tag{4.111}$$

Is a γ-martingale $Z^S(Z^s)^{-1}$ is obviously a P-martingale for all integrable random variable X, we have

$$E^p[X] = E^p[X(Z_T^S)^{-1}Z_T^S] = E^\gamma[X(Z_T^S)^{-1}] \tag{4.112}$$

And so $\frac{dp}{d\gamma} = (Z_T^S)^{-1}$ to be precise p, γ possesses equal measures since their inverse have strictly positive densities.

Example 4.8.1

If $N = d \leq \infty$ with constant coefficients. In this regard the market price is uniquely determine by

$$S_t^i = \frac{\mu_t^i - r_t}{\sigma_t^i}, \quad i = 1, \cdots, d \tag{4.113}$$

And we have $S = \frac{\mu - r}{\sigma}$ by Theorem 4.6.1, the process

$$d\tilde{W}^s := dW_t - dt, \ t \in [0, \ddot{T}]$$

is Brownian with γ measure as shown below

$$\frac{d\gamma}{dp} = E\left(\exp -\tilde{S}W_T - \frac{S^2}{2}T\right) \tag{4.114}$$

Since the exponential martingale has unitary mean, then the evolution of the risky asset is

$$dS_t = rS_t dt + \sigma S_t dW_t^s$$

furthermore the discounted price process $S_t = e^{-rt}S_t$ is a γ-martingale and

$$S_t = e^{-r(T-t)}E^\gamma[S_T|\mathcal{F}_t^{\tilde{W}}]t \in [0, T]$$

Example 4.8.2

If the weight of the risky assets S in the portfolio $\hat{\nu}$ that maximises $H[\eta(X(T))]$ is

$$\hat{\nu}(t) = \frac{\vartheta - \rho}{\sigma^2}$$

at time t if $\rho = 5\%$, $\vartheta = 8\%$, and $\sigma = 20\%$, then an investor should put

$$\frac{0.08 - 0.05}{0.04} = 0.75 \ or 75\%$$

liquid cash on S and 25% riskless asset, at all times. If

$$\eta(x) = x^\gamma$$

then,

$$\hat{\nu}(t) = \frac{\vartheta - r}{(1 - \gamma)\sigma^2}$$

at all times t where $\gamma < 1$ we have that when $\vartheta = 0.2, \ \sigma = 0.3, \ \rho = 0.05, \ \gamma = 0.7$ we have

$$\frac{0.2 - 0.05}{0.3 \times 0.548} = \frac{0.15}{0.1644} = 0.91$$

which implies that 9% of his money is in S, while 91% on risk-free asset. When the risk-aversion is between 2 to 7, the proportion held on contingent claim decreases but, when the risk-aversion is between 0.2 to 1.6,the proportion held in the risky assets increases. The quantity of owned stock reduces as the unwillingness to take risk increases. It make sense to leave a significant amount of money on the risky assets.

It is obvious that the best of portfolios rely largely on the drift of the stock ϑ and the investors risk averse γ. that is, the bigger the value of γ(reluctant in risk), the smaller the Optimal investment on securities. However, if $\vartheta = 80\% = 0.8, r = 75\% = 0.75, \sigma^2 = 10\% = 0.01$ and $\gamma = 90\% = 0.9$ then

$$\frac{0.8 - 0.75}{0.1 \times 0.01} = 50\%$$

Definition 4.8.4

A risky assets S, such that $\mathbb{H}^*_\gamma(S^2) < \infty$ is achievable under a tame,self-funding that is tamed business deal ϕ in the sense that $V^\phi(T) = S$

THEOREM 4.8.2

If any risky assets S satisfies $\mathbb{H}^*_\gamma(S^2) < \infty$ is attainable, then the associated market model is complete. Let S be a risky assets such that $\mathbb{H}^*_\gamma(S^2) < \infty$

Proof:

Let S be a risky asset such that $\mathbb{H}^*_\gamma(S^2) < \infty$ We consider the martingale

$$f(\tau) = \mathbb{H}_{\gamma^\star} e^{-r\xi} S/\alpha_\tau \in [0, \xi] \tag{4.115}$$

By the martingale Representation Theorem,

$$f(\tau) = \mathbb{H}_{\gamma^\star}(f(\tau) + \int_0^a g(\tau) d\tilde{W}_\tau \tau \in [0, \xi] \tag{4.116}$$

Since f is $\gamma^\star$-martingale, we have $\mathbb{H}_{\gamma^\star}(f(\tau)) = f(0)$, the dynamics of the risky assets, S is given by

$$ds(\tau) = r_\tau s_\tau d\tau + \sigma_\tau s_\tau d\tilde{W}_\tau \ \tau \in [0, \xi] \tag{4.117}$$

However,

$$\begin{gathered} dD(\tau) = r_\tau D_\tau d\tau, \\ D(0) = 1 \end{gathered} \tag{4.118}$$

then the strategy

$$\begin{gathered} \phi(\tau) = (\alpha_\tau, \beta_\tau) \\ = \left(f(\tau) - \frac{g(\tau)}{\sigma_\tau}, \frac{g(\tau) D(\tau)}{(\sigma_\tau S_\tau)} \right) \end{gathered} \tag{4.119}$$

is the Tame self-financing strategy that replicates the contingent claim. To see this, observe that the gain process of the strategy is.

$$G^{\phi}_{(a)} = \int_0^a \alpha(\tau) dD(\tau)$$
$$+\int_0^a \beta(\tau)ds(\tau) = \int_0^a \left(f(\tau) - \frac{g(\tau)}{\alpha_\tau}\right) dD(\tau)$$
$$+\int_0^a \frac{g(\tau)D(\tau)}{\sigma_\tau S_\tau} ds(\tau) = \int_0^a f(\tau)dD(\tau) - \int_0^a \frac{g(\tau)}{\alpha_\tau} dD(\tau) + \int_0^a \frac{g(\tau)D(\tau)}{\sigma_\tau S_\tau} r_\tau S_\tau d\tau$$
$$+\int_0^a \frac{g(\tau)D(\tau)}{\sigma_\tau s_\tau}\sigma_\tau S_\tau dW_\tau$$
$$= \int_0^a f(\tau)dD(\tau) - \int_0^a \frac{g(\tau)}{\sigma_\tau} dD(\tau) + \int_0^a \frac{g(\tau)D(\tau)}{\sigma_\tau S_\tau} r_\tau S_\tau d\tau + \int_0^a \frac{g(\tau)D(\tau)}{\sigma_\tau S_\tau}\sigma_\tau S_\tau dW_\tau$$
$$= \int_0^a f(\tau)dD(\tau) - \int_0^a \frac{g(\tau)}{\sigma_\tau} dD(\tau) + \int_0^a \frac{g(\tau)}{\sigma_\tau} dD(\tau) + \int_0^a g(\tau)D(\tau)dW_\tau$$
$$= \int_0^a f(\tau)dD(\tau) + \int_0^a g(\tau)D(\tau)dW_\tau \tag{4.120}$$

Recall that

$$f(\tau) := \mathbb{H}_{\gamma^*}(f(\tau) + \int_0^\tau g(s)d\tilde{W}_\tau \tag{4.121}$$

but $\mathbb{H}_{\gamma^*} f(t) = f(0)$. Thus, substituting (4.116) and (4.117) into (4.119), we have

$$= \int_0^a \left((0) + \int_0^a g(\tau)d\tilde{W}_\tau\right) dD(\tau) + \int_0^a g(\tau)D(\tau)d\tilde{W}_\tau,$$
$$= f(0)\int_0^a dD(\tau) + \int_0^a g(\tau)d\tilde{W}_\tau dD(\tau) + \int_0^a g(\tau)D(\tau)d\tilde{W}_\tau, \tag{4.122}$$
$$= f(0)(D(a) - D(0) + \int_0^a g(\tau)\int_s^a dD(\tau)d\tilde{W}_\tau + \int_0^a g(\tau)B(\tau)d\tilde{W}_\tau$$

By Fubinis

$$= f(0)\left(D(a) - D(0)\right) + \int_0^a g(\tau)\left(D(a) - D(s)\right)d\tilde{W}_\tau + \int_0^a g(\tau)D(\tau)d\tilde{W}_\tau$$
$$= f(0)\left(D(a) - D(0)\right) + D(a)\int_0^a g(\tau)d\tilde{W}_\tau - D(s)\int_0^a g(\tau)d\tilde{W}_\tau + \int_0^a g(\tau)d\tilde{W}_\tau$$
$$= f(0)\left(D(a) - D(0)\right) + D(a)\int_0^a g(\tau)d\tilde{W}_\tau$$
$$= f(0)D(a) - f(0)D(0) + D(a)\int_0^a g(\tau)d\tilde{W}_\tau \tag{4.123}$$

clearly,ϕ is a tame strategy moreover, the value of all the traded assets (or self-financing strategies) (4.122)is determined uniquely by its initial value (4.123) below and must have the same market price

$$V^{\phi}_{(0)} = f(0)D(0) = f(0)$$

Let $M_\tau = D(a)\int_0^a g(\tau)d\tilde{W}_\tau$ $H[M_\tau] = 0$, as $\tau \to \infty$ by convergence theorem Thus M is a martingale i.e.

$$[D(a)\int_0^\tau g(\tau)d\tilde{W}_\tau] = D(a)\int_0^\tau g(\tau)d\tau V^{\phi}_{(\xi)} = f(\xi)D(\xi) = S(\xi) \tag{4.124}$$

Showing that ϕ is self-financing and replicates S. This shows that S is attainable.

4.9 Measures to curb any trace of information asymmetry

This section proffers measures in curbing investors undue advantage of information to have more gains than others in the market: An investor is defined as a dishonest investor haven possessed more than necessary information than others for profit making. We considered the optimal penalty for such investor and the effort of the regulatory agencies to curbed any traces of such. We reveal that the activities of a dishonest investor is curb Biagini *et al.* (2005). It is also observed that as time progresses, the effectiveness of the government and market regulatory bodies as well as onward review of law weight reduces insiders activities. However, a trace of insider trading activity shows the weakness of the regulatory agencies at $\alpha = 1.9$ before the decline at $\alpha = 5$ due to reshuffling of the regulatory agencies staff in avoiding corruption.

4.10 Insider free-market

A space of probability with a filtration $\mathbb{F} := \{\mathcal{F}_t\}$ such that t is between interval $[0, T]$ and for T lies between the interval $\in (0, \infty)$. An investor with a limited resource takes his decisions from the filtration

$$\mathbb{F} := \{\mathcal{F}_t \subset \mathcal{F}\}$$

for $\{0 \le t \le T\}$ resulting from the market, while a sensitive investors information flow is

$$\mathcal{H}_t \subset \mathcal{F},\ 0 \le t \le T$$

and $\mathcal{H}_t \supset \mathcal{F}_t$ A sensitive investors portfolio are stochastic processes adapted to $\mathcal{H}$. Where $\mathcal{H}_t := \mathcal{F}_t \vee \sigma H$, for $t \in [0, T,]$ an enlarged filtration of $\mathbb{F}$ denoting a sensitive investor information flow. An investor with limited resources relied on $\mathcal{F}_t$, while, the sensitive investors information flow is H random variable representing his financial strength, diversification, risk tolerance and future potential price of asset denoted as $\tau \geq T$. It implies the discounted stock process $X = (X_t)_{t \in [0,\tau]}$ value is continuous and $(P, \mathcal{H},)$-semimartingale W with $W_0 = 0$ and a predictable process π such that $\mathbb{E}[\int_0^T \pi_t^2 dW_t] \leq \infty$, then $dS_t = S_t(dW_t + \pi_t dW_t)$ $\forall$ $t \in [0, T]$ and $S_0 > 0$. By Itôs formula, we have $S_t = S_0 e^{\int_0^t (\pi_s - \frac{1}{2}) dW_s + W_t}$ a $\mathbb{H}$-martingale $\tilde{W}_t = W_t - \int_0^t \rho_s^H dW_s,\ t \in [0, T]$. Then from sensitive investors context, the stock price is

$$dS_t = S_t[d\tilde{W}_t + (\rho_t^H + \pi_t) dW_t] \tag{4.125}$$

By Itôs ,

$$S_t = S_0 e^{\int_0^t (\pi_s + \rho_s^H - \frac{1}{2}) dW_s + \tilde{W}_t} \tag{4.126}$$

however, for ordinary investor, a bounded and $\mathcal{F}_t$ -measurable random variable which is duplicated is a complete market. Furthermore, it is the aggregate of a stochastic integral and constant relative to S. Consequently, a measure Q^{ord} of probability identical with P on $(\Omega, \mathcal{F}_\mathcal{T})$ for S a $(Q^{ord}, \mathbb{F})$-martingale is an arbitrage free market.

Definition 4.10.1

An utility function $U : (0, \infty) \to R$ satisfies:

1. continuous differentiability on $(0, \infty)$, strictly concave, and increasing
2. obeys Inadas law $\lim_{j \downarrow 0} \xi'(x) = \infty$ and

$$\lim_{j \to \infty} \xi'(x) = 0 \tag{4.127}$$

   Examples of these are: $U(x) = x^c, 0 < c < 1$ and
   $U(x) = \log(x)$

Definition 4.10.2

A penalty function $f : (0, \infty) \to R$ where f is convex Example is: $f(x) = -cx^{\varphi}$, explain as penalty proportional to flaunting market laws where $c > 0$, $0 < \varphi < 1$ and (x) the utility. Thus let $f(x) = c(x - \tilde{G})^+$ where $(\tilde{G},)$ is the maximum mean gain while C a penalty known as law weight. That is, if an abnormal gains is noticed, he would pay a fine of C.

Definition 4.10.3

set $x > 0$ an initial wealth and $\mathbb{H}$ a general information,then:

1. An $\mathbb{R}$-valued process $\pi = (\pi)_{[0,T]}$ is an $\mathbb{H}$-portfolio if it is an $\mathbb{H}$-adapted predictable process and $\int_0^T \pi_t^2 dW_t$, $pa.s.$

2. $\mathbb{H}$-portfolio π, is self-financing if the discounted wealth process $V(x,\pi)$ agrees with the following equation
$V_t(x,\pi) = x + \int_0^t \pi_s ds_s, \ \forall \ t \in [0,T]$, below $\mathcal{H}_t(\pi) = \int_0^t \pi_s ds_s$ is the gain process

3. an admissible $\mathbb{H}$-portfolio π which is self-financing and
$E[U(V_T(x,\pi)) + (-f)(\mathcal{H}_T(\pi))] < \infty$, where ξ is the utility, and f a penalty function. All the admissible set of portfolios are: $\mathcal{A}_{\mathbb{H}(\mathcal{X},\mathcal{T})}$. changing variable $n_t = \pi_t \frac{S_t}{V_t}$ and the wealth
$V_t(x,n)$. Then
$V_t(x,n) = n_t V_t(x,n)(dS_t)/S_t$. By (4.125)

and It$\hat{o}$'s formula,

$$V_T(x,n) = xe^{\int_0^T n_t d\tilde{W}_t + \int_0^T n_t(\rho_t^H + \pi_t)dW_t - \frac{1}{2}\int_0^T n_t^2 dW_t} \tag{4.128}$$

An ordinary investors problem becomes

$$\max_{n \in \mathcal{A}_{\mathcal{F}}(x,T)} E\left[U(V_T(x,n))\right]$$

while,a sensitive investors problem becomes

$$\max_{n \in \mathcal{A}_{\mathbb{H}}(x,T)} E[U(V_T(x,n)) - f\left(\mathcal{H}_T(n)\right)],$$

then $\xi(x) = \log(x)$ and $f(x) = C_2(x)$,where $C_2 > 0$ is set to be constant known as the severity of the law or penalty and C_1 the probability to be caught and face a punishment, this probability is irrespective of the volume traded. Thus a sensitive investors problem becomes

$$\max_{n \in \mathcal{A}_{\mathcal{H}}(x,T)} E[\log(V_T(x,n)) - C_1 C_2(\mathcal{H}_T(n))]$$

Where we define $\mathcal{H}_T(n) := \mathcal{H}_T \frac{\pi}{v}$ the relative gain process.

Theorem 4.10.1

For $\rho^l \in (\Omega, \mathcal{A}, \mathbb{U})$. Then, $n^{ord}(t) = \pi_t$, is the favorable strategy for uninformed investors and the largest utility is

$$E\left[\log(V_T(x, n^{ord}))\right] = \log x + \frac{1}{2} E\left[\int_0^T \pi_t^2 dW_t\right]. \tag{4.129}$$

while, the fair strategy for a sensitive investors is given by

$$n^s = (1 - C_1 C_2)(\pi_t + \rho_t^H), t \in [0, T]$$

The greatest expected utility is

$$\begin{aligned} &E\left[\log(V_T(x, n^s) - C_1 C_2 \mathcal{H}_T(n^s))\right] \\ &= \log x + \frac{(1 - C_1 C_2)^2}{2} E\left[\int_0^T (\rho_t^{H^2} + \pi_t^2)\right] dW_t \end{aligned} \tag{4.130}$$

α is the government regulatory efficiency. $\alpha C_1 = 1$, represent the probability of observing any abnormal trading but if the Law and enforcement agency is zero $\alpha C_1 = 0$, we have insider trading, this means the regulatory agency strength are ineffective

Proof: from (4.128),

$$\log(V_T(x, n)) = \log x + \int_0^T n_t \, d\tilde{W} \int_0^T n_t(\rho_t^H + \pi_t) dw_t - \frac{1}{2} \int_0^T n_t^2 dW_t$$

and as $\mathcal{H}_T(n) = \int_0^T n_t \frac{(dS_t)}{S_t}$ With (4.125), we have

$$\mathcal{H}_T(n) = \int_0^T n_t d\tilde{W}_t + \int_0^T n_t(\rho_t^H + \pi_t) dW_t$$

then

$$\log((V_T)) - \alpha C_1 C_2 \mathcal{H}_T(n) = \log x + (1 - \alpha C_1 C_2) \int_0^T n_t(\rho_t^H + \pi_t) dW_t$$

If $\alpha C_1 C_2 = 0$, then a sensitive investors highest expected utility is

$$\log(V_T) = \log x + \int_0^T n_t(\rho_t^H + \pi_t) dw_t$$

However, if $\alpha C_1 C_2 = 1$ then this means a sensitive investors information platform changes to $\rho_t^H \subset \rho_t^K$(represented as an undue access to information)Where

$$\mathbb{K}(\mathcal{K}) = \mathcal{K}_t : t \in [0, \infty) \; 0 \leq t \leq T$$

where $\mathcal{K}_t \subset \mathcal{F}$ and

$$\mathcal{K}_t \supset \mathcal{H}_t \supset \mathcal{F}_t$$

Then we have

$$\begin{aligned}\log(V_T(x,n)) - \alpha C_1 C_2 \mathcal{H}_T(n) = \log x + (1-\alpha C_1 C_2)\int_0^T n_t d\tilde{w}_t \\ +(1-\alpha C_1 C_2)\int_0^T n_t(\rho_t^K + \pi_t) dW_t \\ -\frac{1}{2}\int_0^T n_t dW_t\end{aligned} \tag{4.131}$$

and

$$\begin{aligned}\log(V_T(x,n)) - \alpha C_1 C_2 \mathcal{H}_T(n) = \log x + (1-\alpha C_1 C_2)\int_0^T n_t d\tilde{W}_t \\ +\int_0^T \left[((1-\alpha C_1 C_2))(\rho_t^K + \pi_t)\right]^2 dW_t \\ -\frac{1}{2}\int_0^T \left[(1-\alpha C_1 C_2)(\rho_t^K + \pi_t) - n_t\right]^2 dW_t\end{aligned} \tag{4.132}$$

If $E[\int_0^T n_t^2 dW_t] < \infty$ then $\int_0^T n_t dW_t$ is a martingale with zero expectations,

$$\begin{aligned}E\left[\log(V_T(x,n)) - \alpha C_1 C_2 \mathcal{H}_T(n)\right] = \\ \log x + \frac{1}{2}E\left[\int_0^T \left[(1-\alpha C_1 C_2)(\rho_t^K + \pi_T)\right]^2 dW_t\right] \\ -\frac{1}{2}E\left[\int_0^T \left[(1-\alpha C_1 C_2)(\rho_t^K + \pi_t) - n_t\right]^2\right] dW_t\end{aligned} \tag{4.133}$$

In this sequel, we have best expected utility of a sensitive investor to be

$$n^s = (1-\alpha C_1 C_2)(\rho_t^K + \pi_t),\ \forall\, t \in [0,T]$$

thus $n^s \in \mathcal{A}_{\mathbb{K}}$

Remark 4.10.1

Case 1

probability to be caught is zero $\alpha C_1 = 0$, we will have insider transaction that is,

$$n_t^s = (1-\alpha C_1 C_2)(\rho_t^K + \pi_t) = \rho_t^K + \pi_t - \alpha C_1 C_2 \rho_t^K - \alpha C_1 C_2 \pi_t \tag{4.134}$$

Substituting $\alpha C_1 = 0$ into (4.131) we have $n_t^s = \rho_t^K + \pi_t,\ \forall\, t \in [0,T]$ that is, from the maximal expected utility of a sensitive investors equation

$$\begin{aligned}E\left[\log(V_T(x,n^s) - \alpha C_1 C_2 \mathcal{H}_T(n^s))\right] = \log x \\ +\frac{(1-\alpha C_1 C_2)^2}{2}E\left[\int_0^T ((\rho_t^K)^2 + \pi_t^2)\right] dW_t\end{aligned} \tag{4.135}$$

expanding

$$= \frac{1}{2}E\int_0^T \rho_t^{K^2} dW_t - \frac{(\alpha C_1 C_2)^2}{2}E\int_0^T \rho_t^{K^2} dW_t$$
$$+\frac{1}{2}E\int_0^T \pi_t^2 dW_t - \frac{(\alpha C_1 C_2)^2}{2}E\int_0^T \pi_t^2 dW_t \tag{4.136}$$

Setting $C_2 = 0$ and $\rho_t^K = 0$, and similarly from the first integral of (4.132), We have on expansion

$$\frac{1}{2}E\int_0^T \pi_t(1-\alpha C_1 C_2)(\rho_t^K + \pi_t)\times$$
$$(1-\alpha C_1 C_2)(\rho_t^K + \pi_t) dW_t$$
$$= \pi_t[(\rho_t^K + \pi_t - \alpha C_1 C_2 \rho_t^K - \alpha C_1 C_2 \pi_t)\times$$
$$\left((\rho_t^K + \pi_t - \alpha C_1 C_2 \rho_t^K - \alpha C_1 C_2 \pi_t)\right) dW_t \tag{4.137}$$
$$= \pi_t[\rho_t^{K^2} + \rho_t^K \pi_t - \alpha C_1 C_2 \rho_t^{K^2} - \alpha C_1 C_2 \pi_t \rho_t^K +$$
$$\pi_t \rho_t^K + \pi_t^2 \pi - \alpha C_1 C_2 \pi_t \rho_t^K$$
$$-\alpha C_1 C_2 \pi_t^2 + \alpha C_1 C_2 \pi_t \rho_t^K + \alpha C_1 C_2 \pi_t^2]$$

Substituting $C_2 = 0$ and $\rho_t^K = 0$, into (4.134), we have $\frac{1}{2}E\int_0^T \pi_t^2 dW_t$ that is,

$$E\left[\log(V_T(x,\pi) - \alpha C_1 C_2 K_T(\pi))\right] = \log x + \frac{1}{2}E\left[\int_0^T \pi_t^2\right] dW_t$$

setting $\alpha C_1 C_2 K_T(\pi) = 0$ We will have our equation that is, $= E\int_0^T \pi_t^2 dW_t$

Case 2 If the probability to be caught is one that is, $\alpha C_1 = 1$, and if there is a subsequent change in time for the punishment, precisely, $C_2 = \frac{(\rho_t^K)}{(\pi_t + \rho_T^K)}$ we have

$$n^s = n^{ord}$$

Example 4.8.1

From case 1 were α denotes the regulatory agency efficiency in curbing information asymmetry. If $\alpha C_1 = 0$, the probability of catching insider is zero in that case we have insider trading as in case 1, but if $\alpha C_1 = 1$ this means the regulatory agency are effective Given $n^s = (1-\alpha C_1 C_2)(\rho_t^K + \pi_t)$ on expansion we have

$$n^s = \rho_t^K + \pi_t - \alpha C_1 C_2 \rho_t^K - \alpha C_1 C_2 \pi_t \tag{4.138}$$

if $\alpha C_1 = 1$ we substitute into (4.138) we have

$$n^s = \rho_t^K + \pi_t - C_2 \rho_t^K - C_2 \pi_t \tag{4.139}$$

if $C_2 = \frac{(\rho_t^K)}{(\pi_t+\rho_t^K)}$ the penalty weight, we substitute into (4.139)and we have

$$\begin{aligned} n^s &= \rho_t^K + \pi_t - \frac{(\rho_t^K)}{(\pi_t + \rho_t^K)} \times \rho_t^K - \frac{(\rho_t^K)}{(\pi_t + \rho_t^K)} \times \pi_t \\ n^s &= \frac{((\pi_t + \rho_t^K)(\rho_t^K + \pi_t) - \rho_t^{K^2} - \pi_t \rho_t^K)}{(\pi_t + \rho_t^K)} \end{aligned} \tag{4.140}$$

on expansion we have

$$\begin{aligned} n^s &= \frac{(\pi_t \rho_t^K + \pi_t^2 + \rho_t^{K^2} + \pi_t \rho_t^K - \rho_t^{K^2} - \pi_t \rho_t^K)}{(\pi_t + \rho_t^K)} \\ n^s &= \frac{(\pi_t \rho_t^K + \pi_t^2)}{(\rho_t^K + \pi_t)} \rightarrow \frac{(\pi_t(\rho_t^K + \pi_t))}{(\rho_t^K + \pi_t)} \end{aligned} \tag{4.141}$$

thus $n^s = \pi_t$ But $n^{ord} = \pi_t$ (honest investor) thus $n^s = n^{ord}$ You will agree with me that our dishonest investors term $(\rho_t^K + \pi_t)$ has varnished living only honest terms as π_t and this shows that when $\alpha = 1$, there is efficiency of the regulatory agency in the market.

4.11 Simulations

W_t is Brownian given by σW_t and $l = W(T_0), T_0 > T$, where $(\{W_t\})_{\{t \geq 0\}}$ and the drift $\mu(t)$ then,

$$n_t^s = (1 - \alpha C_1 C_2) \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{1}{\sigma(t)} \frac{W(T_0) - W(T)}{T_0 - t}$$

we take

$$C_1(t) = \frac{1 - (T_0 - t)^\alpha}{C_2}, \alpha > 1$$

and $C_2 > 1 - (T_0 - t)^\alpha$, $\forall\ t$ for $T_0 = T < 1$, then a finite Utility is achieve For W_t and $T \leq T_0 \leq 1$, market coefficient $\mu(t) = 0.00090$ and $\sigma(t) = 0.0040\ \forall\ t$ while $r(S) = 0.1$, and the varying-time probability of being punished after a catch $C_1(t) = 1 - (T_0 - t)^\alpha$ where α is the effectiveness of the law. A bigger α means a more effective regulatory agency. while, C_2 is penalty weight. We see several results for honest and dishonest trader, when $\alpha = 1.9$ and $C_2 \in [4, 5, 6]$ we observed that the dishonest trader controls more of his transactions at a higher penalty as well as strong strength of the enforcement agency, at $C_1 \in [0.3845, 0.3885]$ considering a more effective regulatory agency, for example:$\alpha = 5$ with $C_1 \in [0.51, 0.523]$,even with fewer penalties the dishonest traders controls more of his/her trade as shown below on the table: At the regularity condition of $\alpha = 0.9, n_t^{ord} = 19.3125, T_0 = 0.3885, \sigma_t^2 = 0.0016\ t = 0.3845$

$C_1 = 0.2433, C_2 = 4$	$C_1 = 0.9732, C_2 = 5$	$C_1 = 0.1622, C_2 = 6$
$n^s_{t_1} = 58.4539, C_2 = 4$	$n^s_{t_2} = 16.1798, C = 5$	$n^s_{t_3} = 0.5164, C = 6$

Table 4.1: The low efficiency of market regulators

Similarly at regularity of $T_0 = 0.523, \alpha = 5, n_t^{ord} = 19.3125, \sigma_t^2 = 0.0016, t = 0.51$

$C_1 = 0.2499, C_2 = 4$	$C_1 = 0.1999, C_2 = 5$	$C_1 = 0166, C_2 = 6$
$n^s_{t_1} = 7.735 \times 10^{-08}$,	$n^s_{t_2} = 9.66 \times 10^{-08}$,	$n^s_{t_3} = 7.76 \times 10^{-09}$,

Table 4.2: The table indicating the effectiveness of the market regulators

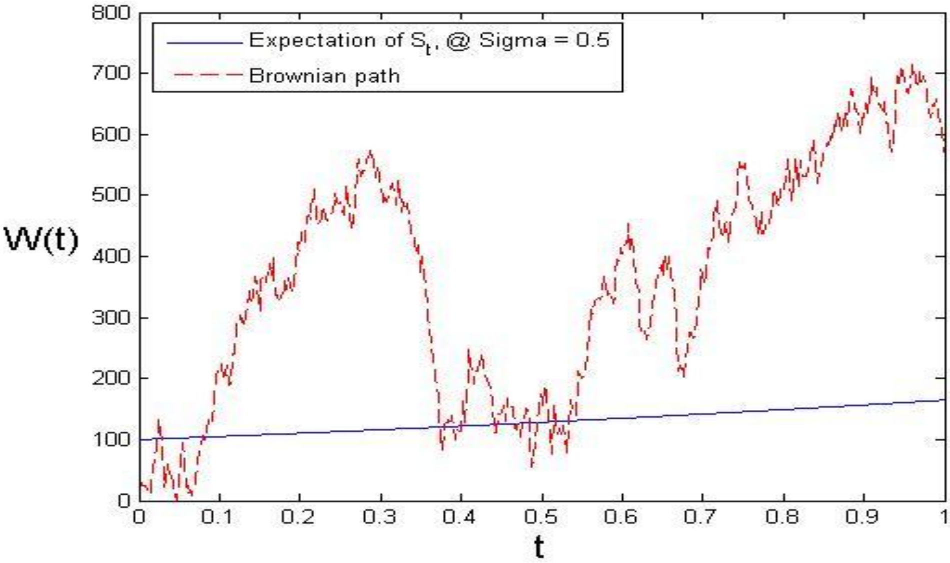


Figure 4.1: The graph of an investor when volatility is 0.5 and drift is 1
Figure represents the low return rate of investor due to inability to take risk capable of generating huge return. It is obvious this investor is at the loss end.

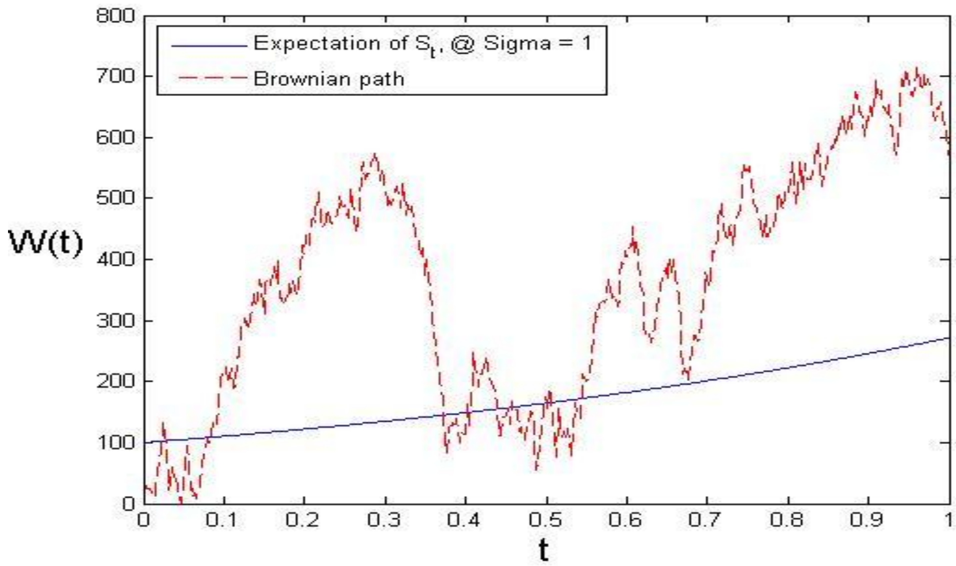


Figure 4.2: The graph of an investor when volatility is 0.8 and drift is 1
Figure 4.2 shows the trend of an investor who's risk appetite is low. On the graph, The investor is not prone to risk, and the part where the investment return outgrown the risk volatility is meager. Such investor goes to the market with the motive of no huge return mentality.

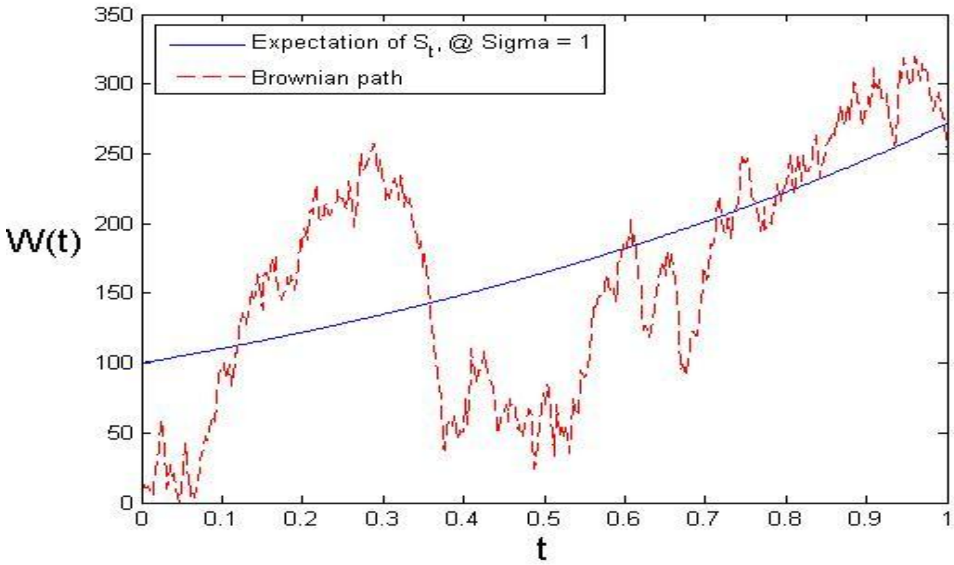


Figure 4.3: The graph of an investor when volatility is 1 and drift is 1

Figure 4.3 shows the result of a sensitive investor who resort to taking more risk by diversifying with the huge rate of return. The trajectory of the return increases significantly and out weight the risk a sensitive investor undergone.

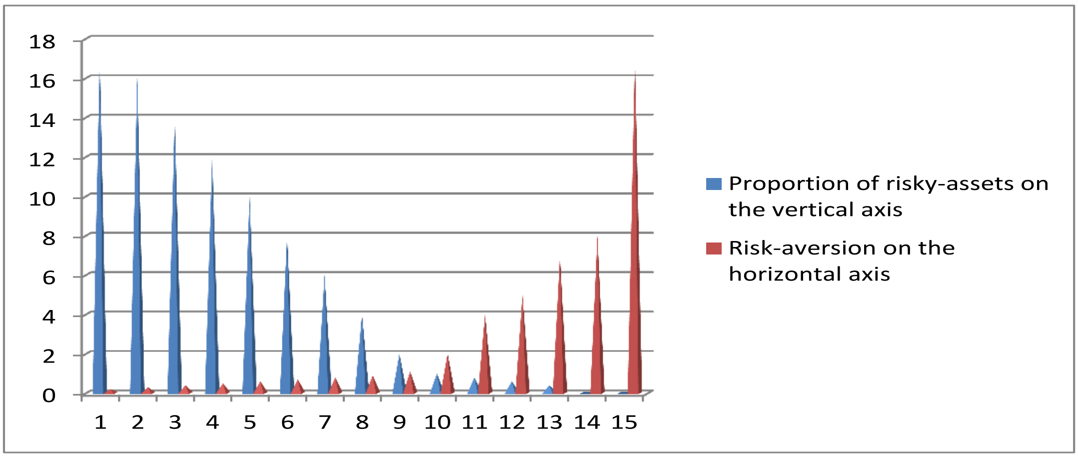


Figure 4.4: This graph indicates the varying rate of risk averse investor

Figure 4.4 indicates the varying rate of risk averse investors behaviour on the risky assets, the proportion of investment on the risky assets at some period is on the increase due to low cost of transaction and positive turn out, At some point in the interval, he refrain from further investing due to high volatility of risky assets.

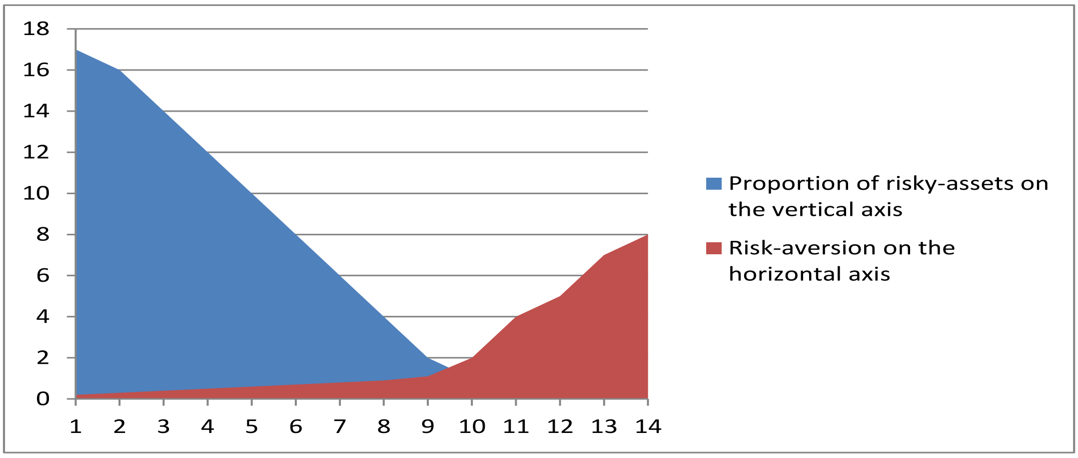


Figure 4.5: This graph indicates huge investment on the risky assets, and lower risk aversion
It is observed that at a lower level of risk-aversion, the proportion held in stock is relatively high to compensate for the decrease on the proportion held in the bond. Similarly, it is equally observed that at a higher level of risk-aversion, the proportion held in stock witnessed a drastic decrease due to higher payoff on the bond

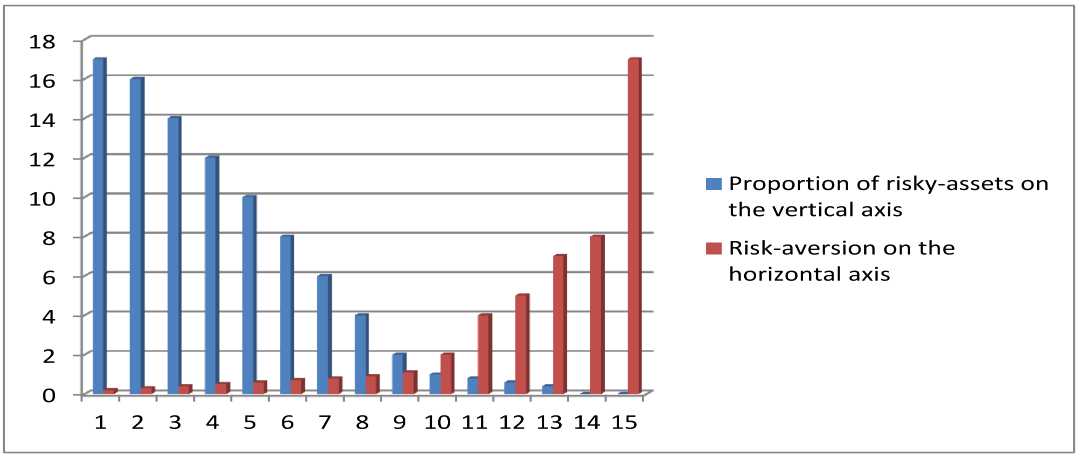


Figure 4.6: This graph indicates the higher risk aversion due to high transaction cost of the risky assets

Figure 4.6 shows the higher risk aversion due to high transaction cost of the risky assets with gradual decrease on the risky assets investment. However, at a specific time, the investment on the risky assets increases due to positive trend in its return.

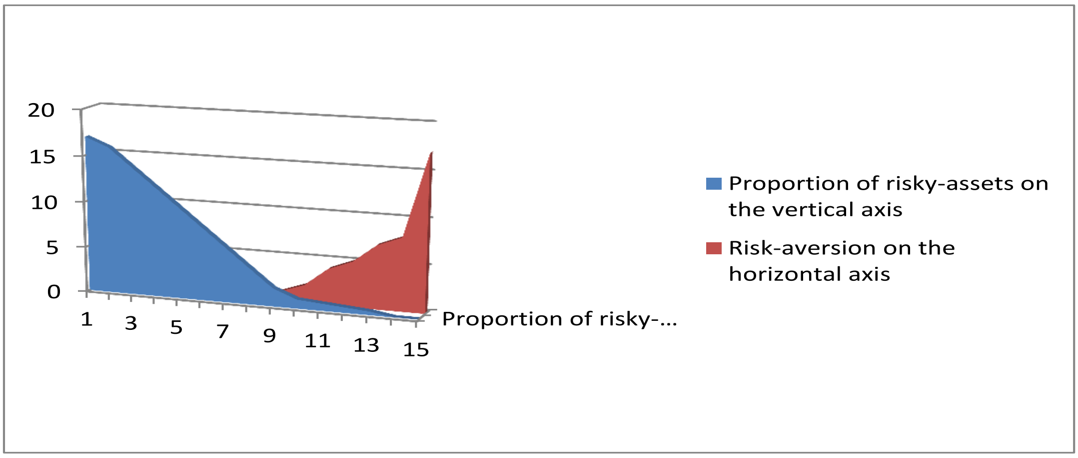
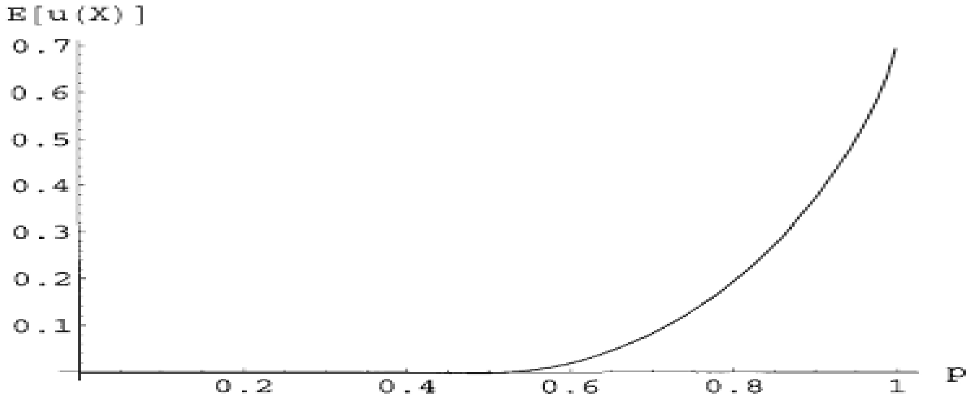


Figure 4.7: Indicating a neutral level of both the risky assets and risk averse
It is observed that at the middle of the risk-aversion, the proportion held on stock and the risk-free assets(bond)are the same showing a risk-free market



6.pdf

Figure 4.8: Indicates the return of the utility function of an investor distributing $100(2p-1)\%$ of his capital in an investment

Figure 4.8 shows the trend of investors possibility for investment, when $\hat{\zeta} = 0.5$ $\alpha = 0$ Thus if $\hat{\zeta} > \frac{1}{2}$ the investor will distribute $100(2\hat{\zeta}-1)\%$ of his capital. that is, when $0 \leq \hat{\zeta} \leq \frac{1}{2}$,then $E[U(X)]$ is the greatest value when $\alpha = 0$, that is, when no investment is made by the investor. But, when $\hat{\zeta} = 0.6 \quad \alpha = 0.2$, similarly when $\hat{\zeta} = 0.7 \quad \alpha = 0.4$, when $\hat{\zeta} = 0.8 \quad \alpha = 0.6$ when $\hat{\zeta} = 0.9 \quad \alpha = 0.8$ the trajectory of the graph increases showing the investors possibility of investing with a positive return.

Chapter 5
SUMMARY AND CONCLUSIONS

5.1 Introduction

Generally, maximising the general expected utility of a sensitive investor in a financial market was studied. A sensitive investor is one who has the financial means and desires to diversify his assets, and seeks for an optimal portfolio which provides the maximum expected returns at a given level of risk.

The present economic situation in Nigeria means that reliance on a single source of income can no longer satisfy the needs of the average middle class family. Thus, the need to explore multiple streams of income is on the front burner of many Nigerian homes. A possible means of achieving multiple streams of income, but an even better means is to invest in an asset and an even better means is to invest in multiple assets. From economics we know that man's needs are insatiable, thus man always seeks for means to increase his expected financial returns.

A sensitive investor seeks and resorts to diversification so as to spread the risk of loss. A diversified portfolio or group of assets has a smoother risk behavior, that is, it is a much more robust investment option Derman (1994).

Diversification aims to reduce the unsystematic risk in portfolio which occur by miss-management, poor forecasting accuracy or wrongful planning processes and decision making. Diversification helps to reduce the volatility of portfolio performance. This is because holding diverse assets implies that the price of diverse assets does not change in the same direction, at the same time or at the same rate. Thus, diversified portfolio is more robust with less variation in expected return.

The optimal portfolio with logarithmic utility fail to consider thorough assessment of the present from the future values of economic influences, therefore, it is short-sighted. However, power utility considers future investment opportunities. For instance, if the dynamics of the current rate of interest of the risky assets would be probably higher than the future, a sensitive investor may consider investing in the risky assets, to its advantage of the potential increase in its price now as a result of its speculated drop in the future. In this respect, we looked at the power utility which maximises expected return of a sensitive investor.

We equally put into consideration of regular checkmating of our portfolio in the sense that combining several business choices of expected zero return into investment with positive expected return is a good measure to achieving desirable portfolio.

An indispensable concept of “buy-low”,sell-high strategy was used in this study for instance, if you owned above 40% of your money in the risky assets, you could transfer some into riskless assets, and if you owned above 60% on the riskless assets, you move some of the capital to risky assets. The significant of this, is to know the best size of money held in the available assets at each trading period. Furthermore, It sounds wise to shuffle your capital between the two assets no matter the size. Our Empirical results show that when volatility $\sigma = 1,\ t = 1,\ S_0 = 100,\ \mu = 1$, the expected return in investment is more than when $\sigma = 0.5$. Moreover if $r = 0.05$, and $\mu = 0.08$ $\sigma = 0.04$ then an investor should consider putting 75% of its money on the risky assets

5.2 Contribution to knowledge

The contribution to knowledge of this study are enumerated below:

1. The optimal portfolio of a sensitive investor was established using power utility function and showed higher investors return as the investor diversify his investment.

2. Two models were derived from the It$\hat{o}$'s integral with respect to power utility function.

3. The extension of the It$\hat{o}$'s integral by forward integral with its lofty properties was used to diversify the investors portfolio.

4. A filtration was built and used as a set of information for the investor.

5. A semimartingale was used to enlarge the investors information.

6. A probability function was defined to capture the activity of an insider in the market and penalty function was established to punish such an insider.

7. A priority Mathematical software was used to compute the investors varying rates of volatility.

5.3 Recommendations

Optimal portfolio of a sensitive investor in a financial market indicates some directions for further research. The Brownian motion was the driving force of the asset price. Thus, an obvious extension is to consider the problem using Lêvy process which initiate the modeling of assets price with jumps. Another possible extension is to consider the variance and correlation of returns on foreign exchange rate and its effect on the utilities and portfolio choice of an investor.

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APPENDICES

Appendix (I) MATLAB code for

```
%BPATH1 Brownian path simulation
clf()
randn('state',100)                          %set the state of randn
T = 1; N = 500; dt = N; sigma = 0.5;
dW = zeros(1,N);                             %preallocats arrays ...
W = zeros(1,N);                              % for efficiency
t = [0:0.002:1];
E = 100*exp(sigma*t);
dW(1)= sqrt(dt)*randn;  %first approximation outside the loop ...
W(1)= dW(1);             %since W(0) = 0 is not allowed
for j = 2:N
   dW(j)= sqrt(dt)*randn;          % general increment
   W(j)= abs(W(j-1) + dW(j));
end
plot([0,t], [1,E],'b-')
hold on
plot(t, [1,W], 'r--')                        %plot W against t
hold off
%xlabel('t', 'Fontsize', 16)
%ylabel('W(t)','Fontsize',16,'Rotation',0)
xlabel('t','FontSize',16)
ylabel'W(t)','FontSize',16,'Rotation',0,'HorizontalAlignment',\\
'right'
legend('Expectation of S_t, @ Sigma = 0.5','Brownian path',2)
```

Appendix (II) MATLAB code for

```
%BPATH1 Brownian path simulation
clf()
randn('state',100)                         %set the state of randn
T = 1; N = 500; dt = N; sigma = 1;
dW = zeros(1,N);                           %preallocats arrays ...
W = zeros(1,N);                            % for efficiency
t = [0:0.002:1];
E = 100*exp(sigma*t);
dW(1)= sqrt(dt)*randn;  %first approximation outside the loop ...
W(1)= dW(1);            %since W(0) = 0 is not allowed
for j = 2:N
   dW(j)= sqrt(dt)*randn;      % general increment
   W(j)= abs(W(j-1) + dW(j));
end
plot([0,t], [1,E],'b-')
hold on
plot(t, [1,W], 'r--')               %plot W against t
hold off
%xlabel('t', 'Fontsize', 16)
%ylabel('W(t)','Fontsize',16,'Rotation',0)
xlabel('t','FontSize',16)
ylabel'W(t)','FontSize',16,'Rotation',0,'HorizontalAlignment',\\
'right'
legend('Expectation of S_t, @ Sigma = 1','Brownian path',2)
```