BRINKMAN-FORCHHEIMER EQUATIONS IN HOMOGENEOUS SOBOLEV SPACES

BY

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Certification

I certify that this work was carried out by Kabiru Michael ADEYEMO with matriculation number 128297 in the Department of Mathematics, Faculty of Science, University of Ibadan under my supervision

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Dedication

This thesis is dedicated to the Lord of hosts, the Author and Finisher of my faith.

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Abstract

Brinkman-Forchheimer Equations (BFE) are used to describe non-Darcy behavior of fluid flow in a saturated porous medium. Researchers in this area have focused on the study of structural stability and long time behavior of solutions in square integrable space L^2. However, problems on existence of weak solutions in the homogeneous Sobolev space $\dot{H}^s$ and stability of the system in the critical homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}$ remain relatively unresolved. This study was therefore designed to obtain existence of weak solutions in $\dot{H}^s$ and stability of the system with respect to initial data, φ_n in $\dot{H}^{\frac{1}{2}}$.

Galerkin method was employed to obtain weak solutions in $\dot{H}^s = \{u \in S'(\mathbb{R}^3) : ||u||_{\dot{H}^s(\mathbb{R}^3)} < +\infty\}$ where S' is dual space of tempered distribution and $||u||$ is the norm of u. BFE was approximated by finite-dimensional problem when the damping term is continuous, continuously differentiable and satisfies Lipschitz condition. Cauchy-Schwarz inequality and Parseval identity were used to obtain uniform bounds on the Fourier transforms of some terms in L^2 appearing in the weak formulation. The concept of profile decomposition was employed to show that the sequence of solutions of BFE associated with sequence of initial data was bounded in $E_\infty = C_b^0((\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ where $\dot{H}^{\frac{3}{2}}$, is a space of vector fields whose first derivative is in $\dot{H}^{\frac{1}{2}}$, C_b^0, is a space of bounded continuous functions and L^4 is a space of four times integrable functions. By using the orthogonality property of sequence of scales and cores, (h_j, x_n) in $(R^+ \setminus \{0\} \times \mathbb{R}^3)$, the sequence of solutions was decomposed into a sum of orthogonal profiles in E_∞ to generate a priori estimate. Finite time singularities of solutions was also obtained with respect to singularity generating initial data by profile decomposition.

The weak solution $u(x,t)$ obtained belongs to $L^\infty((R^+, \dot{H}^s(R^3)) \cap L^2(R^+, \dot{H}^{s+1}(R^3)) \cap L^{r+1}(R^+, L^{r+1}(R^3))$ for $r \geq 1$ when the damping term is continuous and satisfies the estimate $Sup_{0\leq t \leq T}||u||_{\dot{H}^s(R^3} + 2\lambda \int_0^T ||\bigtriangledown u||^2_{\dot{H}^s(\mathbb{R}^3)} dt + 2\beta \int_0^T ||u||^{r+1}_{L^{r+1}} dt \leq ||u_0||^2_{\dot{H}^s}$

where λ and β are positive constants. Also for continuously differentiable biharmonic damping term, the solution is in $L^{\infty}((R^{+}, L^{2}(R^{3})) \cap L^{2}(R^{+}, \dot{H}^{1}(R^{3})) \cap L^{2}(R^{+}, \dot{H}^{2}(R^{3}))$ and satisfies $Sup_{0\leq t\leq T}||u||^{2}_{L^{2}} + 2\lambda\int_{0}^{T} || \bigtriangledown u||^{2}_{L^{2}}dt + 2\beta\int_{0}^{T} ||u||^{2}_{H^{2}} \leq ||u_{0}||^{2}_{L^{2}}$. The damping term with Lipschitz condition satisfies the estimate $Sup_{0\leq t\leq T}||u||^{2}_{\dot{H}^{s}(R^{3})} + 2\lambda\int_{0}^{T} || \bigtriangledown u||^{2}_{\dot{H}^{s}(R^{3})}dt + 2\beta\int_{0}^{T} ||u||^{r+1}_{L^{r+1}} \leq ||u_{0}||^{2}_{\dot{H}^{s}(R^{3})}$ for $u \in L^{\infty}((R^{+}, \dot{H}^{s}(R^{3})) \cap L^{2}(R^{+}, \dot{H}^{s+1}(R^{3})) \cap L^{r+1}(R^{+}, L^{r+1}(R^{3}))$ and $r \geq 1$ where L^{∞} is a space of essentially bounded functions. The sequence of solutions, u_{n} obtained was bounded in E_{∞}. The boundedness of the sequence of solution was due to the existence of a bound $||\varphi_{n}|| \leq \rho$ on the sequence of initial data where ρ is any real number in $[0, C^{A}_{BFE}]$. Then a priori estimate $||BFE(\varphi)||_{E_{\infty}} \leq B(||\varphi||_{\dot{H}^{\frac{1}{2}}(R^{3})}, ||u||_{A})$ was obtained where B, $BFE(\varphi_{n})$, A, C^{A}_{BFE} are non-decreasing function from $\mathbb{R}^{+} \times [0, C^{A}_{BFE}]$ to $\mathbb{R}^{+}$, solution of BFE associated with initial data φ, admissible space, constant in $R^{+} \cup \{+\infty\}$ respectively. Existence of singularity generating initial data gave rise to finite time singularities of solution that did not belong to E_{∞}.

The existence of weak solutions was obtained and the system was found to be stable with respect to initial data in the homogeneous Sobolev space.

Keywords : Stability, Singularity generating initial data, Critical space, Porous medium, Darcy behavior

Word count: 491

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The following notations were used in this work

L^2 denotes space of square integrable functions

$\dot{H}^s$ denotes homogeneous Sobolev space of order s

φ denotes initial data

S' denotes the dual space of tempered distribution

C_b^0 denotes space of bounded continuous functions

L^4 denotes the space of four times integrable functions

L^∞ denotes the space of essentially bounded functions

nu and β are Brinkman and Forchheimer coefficients respectively

$\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ denotes homogeneous Besov space

$||u||$ denotes the norm of the function u

B_{BF}^A denotes the largest open ball in the admissible space 'A' with centre zero

Y^* is the dual of the Banach space Y

$\hat{u}$ denotes the Fourier transform of function u

$BF(\varphi)$ denotes solution of BFE associated with initial data φ

(h_n, x_n) denotes sequence of scales and cores

Chapter 1
INTRODUCTION

1.1 Preliminary Considerations

Many occurrences and processes in natural sciences can be explained by means of partial differential equations. The common questions arising from these equations are ones of solvability, stability, regularity, existence and approximation of solutions. Many complex dynamical systems can be represented by a small number of short equations. In general, it is not simple to find solutions to them. In fluid mechanics, Navier-Stokes equation (NSE) which describes the motion of a Newtonian fluid, is a good example for this. The development of basic dynamic laws like Newton's laws of motion as well as Newton's laws of viscosity, Pascal's law etc led to the formation of these equations. The NSE is a model used to describe the motion of a fluid which is incompressible and viscous, for instance; water, oil, glycerine, and also air. The equations are used in many branches of applied sciences, which include Meteorology, Oil Industry, Geology, Ship and Car Industries, Airplane, Biology and Medicine.

However, being a very useful equation does not imply that they can model correctly any fluid under any situation. In fact, the areas where they can be applied is limited to the Newtonian fluids. This area includes liquids like water, aqueous solutions, salt solutions, motor oil, gasoline, mineral oil and kerosene. Description of fluid is also of interest in fluid mechanics depending on factors like temperature, pressure, external forces, or properties of the fluid itself. When external forces are fixed, the nature of a fluid and the corresponding model is determined by the behavior of its internal stress. If the linear relation is assumed between the strain and the stress, i.e assuming a constant viscosity which reflects in results involving Navier-Stokes equations and the description of so called Newtonian fluids like water for example but there are many fluids that exhibit 'unusual' behavior and cannot be modeled with these equations. Those fluids are called non-Newtonian. They appear in nature, e.g blood, lava, mud slurries or snow, in our everyday life, e.g toothpaste, paints or tomato ketchup as well as in many industrial environments, polymetric liquids or melts. In order to model non-Newtonian fluids, one has to derive suitable constitutive laws for the internal stress which means relations between the internal stress, strain, external forces, etc.

Non-Newtonian fluid mechanics is a vast subject, that has gained much attention in mathematics, biology, chemical engineering and geophysics. Many different models incorporating the several phenomena exhibited by non-Newtonian fluids have been introduced and investigated. Non-Newtonian fluid phenomena is one of the deviation situations from Darcy's law in nature.

1.1.1 Darcy law

In 1856, Henry Darcy proposed a simple model for an incompressible fluid flow in rigid porous media and states that the (Darcy) velocity and pressure gradient are linearly proportional to each other. He empirically obtained the model which is based on the experiments performed on the flow of water in sand beds. The equation has been used in diverse fields (e.g petroleum engineering, civil engineering, polymer engineering), and in various technological applications like enhanced oil recovery, designing filters, and geological sequestration of carbon-dioxide .

Darcy flow model represents a linear relationship between the pressure gradient and flow rate in porous media.

$$u_f = -\frac{k}{\mu}\nabla p, \tag{1.1.1}$$

where u_f, k, μ and p are the velocity, permeability of the medium, fluid viscosity, and the pressure respectively. Otherwise, it is non-Darcy flow. The equation is essentially founded on empirical evidence and cannot be derived analytically by performing a momentum balance on a small element of the porous medium. In recent years however, it has been shown that Darcy's model can be achieved numerically in two ways: The first method is application of volume averaging theory on the NSE, and the other is using mixture theory (also known as the theory of interacting continua). The resulting constitutive model can then be discretised and used to obtain numerical solutions for problems related to heterogeneous flow through porous media.

1.1.2 Limitations of Darcy model and its generalizations

Darcy equation is simply an approximation of the linear momentum in the context of theory of interacting continua. Also it merely predicts flux but cannot predict stresses in solids. The Darcy model was formulated based on the following assumptions:

(i) No mass production (i.e no chemical reactions)

(ii) The porous solid is assumed to be rigid. Thus, the balance laws for the solid are satisfied. In particular, the stresses in the solid are what they need to ensure that the balance of linear momentum is met.

(iii) The fluid is viscous, steady, incompressible and homogeneous. There is also an assumption on velocity and its gradient to be small in order to ignore inertial effects.

(iv) Dissipation of energy does not occur between fluid layers.

(v) The only interaction force is at the fluid and pore boundaries.

According to Darcy's model, the discharge flux is a function of drag coefficient (which has value as viscosity over permeability) and pressure gradient. The problem is that it is independent of pressure and viscosity, thus assuming drag coefficient to be constant. Experimental studies have shown that it is not constant and differs with pressure and/or velocity. Therefore, Darcy's model as it is may not give us accurate results of flow because it does not take into account pressure and velocity dependence- huge setbacks such as overly estimated discharge fluxes or inaccurate pressure contours may arise if used appropriately. Roughly speaking, this law does not take into account the acceleration forces or inertia in the fluid in comparison to the classical NSE.

There are many situations where nature does not agree with Darcy law, for example a situation involving high velocity, ionic and molecular effects or when there is a presence of some fluids phenomena that are non-Newtonian . In situations like these, it is important to develop a more adequate model that will not be based on the law. In practice, this can be possible by employing Forchhemer equation which is a result of an observation of P. Forchheimer and states that the connection between the flow rate as well as pressure gradient is nonlinear with high velocity and that this nonlinearity increases with flow rate. Darcy-Forchheimer law states

$$\nabla p = -\frac{\mu}{k} u_f - \gamma \rho_f |u_f|^2 u_f \tag{1.1.2}$$

Where $\gamma > 0$ is the so called Forchheimer coefficient and u_f stands for the Forchheimer velocity and ρ_f the density.

Brinkman equation is an equation between Darcy's equation and the NSE, where the viscous term is added to Darcy's equation or inertial term is dropped from NSE.

Recently, research works in porous media utilize Brinkman-Forchheimer equation (BFE).

$$(BFE) \left\{ \begin{array}{rcll} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p + \beta |u|^{r-1} u & = & 0 & \text{in } \mathbb{R}^+ \times R^3 \\ \nabla \cdot u & = & 0 & \\ u \mid_{t=0} & = & u_0 & \end{array} \right. \tag{1.1.3}$$

The BFE is based on Darcy-Forchheimer law and it was first derived classically when ($r = 2,\ \beta > 0$) in the framework of thermal dispersion using volume averaging method of the temperature deviations and velocity in the pores. $u(x,t)$, $p(x,t)$ and ν are the velocity field, the scalar function pressure and a constant (Brinkman coefficient i.e effective viscosity) respectively and β denotes Forchheimer coefficient.

The Brinkman-Forchheimer equation is suitable to describe the motion of incompressible fluid flows . This model is used in tidal dynamics and in the theory of non-Newtonian fluids. These can be found in the work of (Zao and You, 2012), (Kalantarov and Zelk, 2012) and (Roccer and Zang, 2012). However, its application is restricted to flows with high velocities and when the porosities are not too small. That is when Darcy law fails to hold for a porous medium.

1.1.3 Historical setup for theory of profiles

Gerard P.(1998) introduced the theory of profiles in order to give a description of the compactness defect of the Sobolev embeddings. It was used by H. Bahhouri and Gerard P. (1999) (for the critical 3D wave equation) to investigate subtil properties of the quintic wave equation on $\mathbb{R}^3$. The ideas of Bahhouri H. and Gerard P. (1999) were revisited by Keraami S. (2001) and. Gallagher Iftimie I. and Planchon F.

(2013) for the Schrodinger and NSE to give a description of the structure of the sequences for the solutions that are bounded to those equations. The techniques have been used successfully so as to investigate blow-up of solutions of nonlinear pdes. For instance, Kenig C.E. and Merle F.(2008) worked on scattering, blow-up and well-posedness for the critical nonlinear wave equation, Rusin W. and Sverak V. (2013) studied initial data in L^3 for possible singularities of NSE. Gallaghher I., Koch G. and Planchon F. (2013) also considered the method in setting of $L_t^\infty(L_x^3)$ NSE regularity criterion.

1.1.4 Solutions of Brinkman-Forchheimer equations

a **Weak solutions**:

The function $u(t,x)$ is called a weak solution of BFE if $T > 0$, the three conditions are satisfied as follows:

(i) u is in $L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \cap L^{r+1}(0,T;L^{r+1}(\mathbb{R}^3))$

(ii) Given any $\phi \in C_c^\infty([0,T) \times \mathbb{R}^3)$ and $\mathrm{div}\phi = 0$,

$$-\int_0^T \int_{\mathbb{R}^3} u\phi_\tau dxdt + \nu \int_0^T \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \partial_j u_i \partial_j \phi_i dxdt + \int_0^T \int_{\mathbb{R}^3} (u \cdot \nabla u + \beta |u|^{r+1} u)\phi dxdt = \int_{\mathbb{R}^3} u_0 \phi(0,x) dx \quad (1.1.4)$$

(iii) $\mathrm{div} u = 0$ a. e $(t,x) \in (0,T) \times \mathbb{R}^3$

The weak solution u of BFE is global if for all $T > 0$, it is a weak solution

b **A Leray-Hopf weak solution** of the BFE with the initial data $u_0 \in H$ is a weak one if the energy inequality is satisfied and for $r = 3$, we have:

$$||u(t_1)||^2 + 2\nu \int_{t_0}^{t_1} ||\nabla u(s)||^2 ds + 2\beta \int_{t_0}^{t_1} ||u(s)||_{L^4}^4 ds \leq ||u(t_0)||^2 \quad (1.1.5)$$

c **Mild solutions:**

Consider a Banach space $X \subset S'(\mathbb{R}^3)$ and let $Y_T \subset L^2_{loc}((0,T) \times \mathbb{R}^3)$ be $\ni$ if $\phi \in X$ then $S(t)\phi(x) \in Y_T$ for $0 < t < T$. Let $\phi(x) \in X$ be such that $\mathrm{div}\phi = 0$. We call $u(t,x) \in Y_T$ a mild solution on $(0,T)$ when it satisfies the integral identity

$$u(t,x) = S(t)\phi(x) - \int_0^t S(t-s)P\text{div}(u \otimes u)(s)ds \tag{1.1.6}$$

d **Strong solutions**

The function $u(t,x)$ is a strong solution of BFE on $(0,T) \times \mathbb{R}^3$ if it is a weak solution and satisfies

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)) \cap L^\infty(0,T;L^{r+1}(\mathbb{R}^3))$$

1.1.5 Some Fundamental Definitions, Spaces and Results

The mathematical theory of Brinkman-Forchheimer equations is based on the usage of some function spaces which are crucial in the modern theory of pdes. In this section, we give definitions of some function spaces. Homogeneous Sobolev spaces are given a particular interest. Related inequalities with these function spaces are also stated. Basic definitions of some terms are itemized and few results are stated.

a **Weak Convergence**

Steps to solve pdes normally entail a sequence of solutions to approximate problems. Uniform bounds can easily be gotten for the approximate solutions in some Banach space, but to show convergence in the Banach space is an herculean task. In a situation like this, weak convergence is adopted. It means that when some linear functionals are applied to the sequence, they will yield convergent sequences. The definition is as follows:

Let Y be a Banach space. A sequence $y_n \in Y$ converges weakly to y if $g(y_n)$ converges to $g(y)$ for every $g \in Y^*$. A sequence g_n in Y^* converges weakly-$*$ to g if $g_n(y)$ converges to $g(y)$ for every $y \in Y$.

we write $y_n \longrightarrow y$ for convergence in norm, $y_n \rightharpoonup y$ for weak convergence and $y_n \rightharpoonup^* y$ for weak-$*$ convergence.

b **Dual space X'**

The set of all continuous, linear functionals on X is called the dual of X and is denoted by X'

c **Convergence in the norm space Y**

$\{y_n\}$ in Y is convergent to y_0 if and only if $\lim_{n\to\infty} ||y_n - y_0||_Y = 0$ in $\mathbb{R}$.

d **Separable space:**

$K \subset Y$ is said to be dense in Y if each $y \in Y$ is the limit of a sequence of members of K. The normed space Y is called separable if it has a countable dense subset.

e **Banach space:**

A sequence $\{y_n\}$ in Y is Cauchy if and only if $\lim_{m,n\to\infty} ||y_m - y_n||_Y = 0$. If very Cauchy sequence in Y converges to a limit in Y, then Y is complete and a Banach space.

f **Open ball:**

If $r > 0$, the set $B_r(x) = \{y \in X : ||y - x||_X < r\}$ is called the open ball of radius r with center $x \in X$

g **Compact sets**

A subset B of a normed space Y is called compact if every sequence of points in B has a subsequence converging in Y to an element of B. Compact sets are closed and bounded, but closed and bounded sets need not be compact unless Y is finite dimensional.

h **Coercivity**

A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is coercive if $\frac{f(u)\cdot u}{|u|} \to \infty$ as $|u| \to \infty$

i **Continuity**

A mapping $F : Y \to Y^*$ is bounded if it maps bounded sets in Y to bounded sets in Y^*. The mapping is continuous if for every $u \in Y$ we have $||F(u) - F(v)||_{Y^*} \to 0$ whenever $||u - v||_Y \to 0$

j **Monotone mapping**

A mapping $F : Y \to Y^*$ is monotone if $(F(u) - F(v), u - v) \geq 0$ for all $u, v \in Y$

- **Lebesgue spaces L^p:**

  For $p \in [1, \infty)$, and $L^p = L^p(\Omega)$ the space of all functions u (real) defined in $\Omega \ni$

$$||u||_p \equiv \left(\int_\Omega |u|^p \right)^{\frac{1}{p}} < \infty$$

The functional defines a norm in L^p, which makes L^p becomes a Banach space.

In summary

Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq q < \infty$. If $u : \Omega \to \mathbb{R}^n$ is measurable, we define

$$||u||_{L^q} = \begin{cases} \left(\int_\Omega |u|^q \mathrm{d}x \right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty \\ \\ \underset{\Omega}{\text{esssup}} |u| & \text{if } q = \infty \end{cases}$$

For $q = 2$, L^q is Hilbert under $\langle u, v \rangle \equiv \int_\Omega uv, u, \; v \in L^2$. To this scalar product corresponding to the norm

$$||u|| = \left(\int |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

For $1 \leq p \leq \infty$ is set

$$q' = \frac{p}{(p-1)}$$

then the Holder inequality follows

$$\int |uv| \leq ||u||_p ||v||_{p'}$$

$\forall \; u \in L^p(\Omega), \; v \in L^{p'}(\Omega)$ (Miranda, 1978). p' is the Holder conjugate of p. The inequality shows that the bilinear form $\langle u, v \rangle$ whenever $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$. In case $q = 2$, the inner product and the corresponding norm are related by Cauchy-Schwarz inequality.

$$|\langle u, v \rangle| = \left| \int_\Omega u(x)v(x)dx \right| \leq \left(\int_\Omega |u(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |v(x)|^2 dx \right)^{\frac{1}{2}}$$

. For all $u, v \in L^2$. Generally,

$$\int_\Omega |u_1 u_2 \cdots u_m| \leq ||u_1||_{p^1} ||u_2||_{p^2} \cdots ||u_m||_{p^m},$$

where

$$u_1 \in L^{p1}(\Omega), \; u_2 \in L^{p2}(\Omega), \cdots, u_m \in L^{p^m}(\Omega)$$

and

$$\sum_{i=1}^{m} q^{-1} = 1, 1 \leq p_i \leq \infty, i = 1, ..., m.$$

Both inequalities are the result of Young inequality:

$$ab \leq \frac{\epsilon a^q}{q} + \epsilon^{\frac{-q'}{q}} \frac{b^{q'}}{q'} (a, b, \epsilon > 0) \tag{1.1.7}$$

$\forall \; p \in (1, \infty)$. When $p = 2$, the relation is the Cauchy inequality.

Holder inequality also results to the Minkowski inequality:

$$||u + v||_q \leq ||u||_q + ||v||_q, u, v \in L^q(\Omega) \tag{1.1.8}$$

$$||u||_q \leq ||u||_s^{\theta} ||u||_r^{1-\theta} \tag{1.1.9}$$

is an interpolation inequality valid for all

$$u \in L^s(\Omega) \cap L^r(\Omega)$$

with

$$1 \leq s \leq q \leq r \leq \infty,$$

and

$$q^{-1} = \theta s^{-1} + (1-\theta) r^{-1} \; \theta \in [0,1]$$

- **Spaces $W^{m,p}$:**

The notation $W^{m,p}(\Omega)$ is the Sobolev space of differentiability m and integrability p. It contains functions which are k-weakly differentiable and $D^{\alpha}u \in L^p(\Omega)$ for all $|\alpha| \leq k$

$$W^{m,p} = \{u \in L^p; \; D^{\alpha}u \in L^p \text{ for all } \alpha, \; |\alpha| \leq m\}$$

The Sobolev space

$$H^m = W^{m,2}$$

$$W^{m,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \; \xi^{\alpha}\hat{u} \in L^2(\mathbb{R}^n) \; \forall \; \alpha, \; |\alpha| \leq m\}$$

$$||u||_{m,2}^2 = ||u||_{W^{m,2}}^2 = \sum_{|\alpha| \leq m} ||\xi^{\alpha}\hat{u}||_{L^2}^2$$

The Homogenous Sobolev space $\dot{H}^s(\mathbb{R}^3)$

$\dot{H}^s(\mathbb{R}^3)$ is the homogeneous Sobolev space of order s, defined

$$\dot{H}^s(\mathbb{R}^3) = \{u \in S'(\mathbb{R}^3); \; ||u||_{\dot{H}^s(\mathbb{R}^3)} < +\infty\},$$

where

$$||u||_{\dot{H}^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}},$$

and $\hat{u}$ denotes the Fourier transform of u. The inhomogeneous Sobolev space $H^s(\mathbb{R}^3)$ is defined the same way by replacing $|\xi|^{2s}$ with $(1+|\xi|^2)^s$

Schwartz class, $S(\mathbb{R}^n)$, is the vector space of functions which are C^{∞} and which, as well as all their derivatives, decay faster than any polynomial rate. If $f \in S$, then given any $\alpha, \; \beta \in N_0^n$, $\exists \; C_{\alpha,\beta}$ such that

$$Sup_{x \in \mathbb{R}^n} |x^{\alpha} \partial_x^{\beta} f(x)| \leq C_{\alpha,\beta}$$

- **Besov space:**

The space, $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ stands for homogeneous Besov space. Elements of

$\dot{B}^s_{p,\infty}(\mathbb{R}^3)$ satisfy

$$||u||_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} = sup_{j\in\mathbb{Z}}2^{js}||\Delta||_j u||_{L^p(\mathbb{R}^3)} < +\infty$$

- **Fourier transform** Let $u \in L^1(\mathbb{R})$, then the function $\hat{u}$ defined as
$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-ix\xi}u(x)\mathrm{d}x$$
is called the Fourier transform of u. Its inverse Fourier transform is defined as
$$\check{u}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{ix\xi}u(\xi)\mathrm{d}\xi$$

Theorem 1.1.4 (Weak compactness, Alaoglu):

Let Y be a separable Banach space and let g_n be a bounded sequence in Y^*. Then g_n has a weakly-$*$ convergent subsequence.

Theorem 1.1.5 (Robert A. Adams, Sobolev spaces)

$L^p(\Omega)$ is separable if $1 \leq p < \infty$

Theorem 1.1.6 (Robert A. Adams, Sobolev spaces)

C_0^∞ is dense in $L^p(\Omega)$ if $1 \leq p < \infty$

Theorem 1.1.7 (Robert A. Adams, Sobolev spaces)

$W^{m,p}$ is separable if $1 \leq p < \infty$ and is reflexive and uniformly convex if $1 < p < \infty$. In particular, therefore, $W^{m,2}(\Omega)$ is a separable Hilbert space

1.2 Auxiliary result

The existence result by T. Kato and H. Fujita for Navier-Stokes equation (NSE) is very useful in this study. The existence of solution of NSE in the homogeneous Sobolev space was discussed in their work.

Theorem 1.2.1 (Fujita & Kato, 1962))

If $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then there exists a unique maximal time $T_* > 0$ and a solution u of (NSE) which is unique associated with u_0 such that

$$u \in C^0\left([0,T], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T], \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \text{ for all } T < T_*$$

Moreover, if $T_* < +\infty$, then we have

$$\lim_{T\to T_*} ||u||_{L^2([0,T],\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} = +\infty$$

More so, a constant C_E exists such that

$$||u_0||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq C_E \implies T_* = +\infty$$

and for $t \geq 0$,

$$||u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \int_0^t ||u(s)||^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} ds \leq ||u_0||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \tag{1.2.1}$$

One can think of C_E as a bound on the initial data such that a solution is obtained together with the energy estimate (1.2.1), this of course implies that $T_*(\varphi) = +\infty$. Gallagher I.(2001), in her paper, considered a decomposition of sequence of initial data bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and analyzed how this decomposition is propagated by the Navier-Stokes equations.

1.3 Objectives of the study

The objectives of this work are as follows:

(i) To obtain existence results for weak solutions of Brinkman-Forchheimer equations in a homogeneous Sobolev space when the damping term $f(u)$ is continuous, continuously differentiable and satisfies Lipschitz continuity condition.

(ii) To analyze how Profile Decomposition is propagated by Brinkman-Forchheimer equations.

(iii) To obtain stability results for the global solutions of Brinkman-Forchheimer equations in a critical space.

(iv) To investigate the possibility of singular solutions for BFE in a critical Sobolev space.

1.4 Statement of the problem

We consider homogeneous, three dimensional Brinkman-Forchheimer equation

$$(BFE) \begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + f(u) &= 0 \quad \text{in } \mathbb{R}^+ \times R^3 \\ \nabla \cdot u &= 0 \\ u\,|_{t=0} &= u_0 \end{cases} \tag{1.4.1}$$

where $u(t;x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity, $p(t;x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ is the associated pressure and $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is given divergence-free initial data, the constant ν is the positive Brinkman coefficient (effective viscosity). The divergence free condition is for the fluid incompressibility. The research so far in the literature have been looking at the behavior of solution of BFE in square integrable space L^2 and other functional spaces without really exploring the homogeneous Sobolev spaces. Considering the equation in homogeneous Sobolev spaces will make possible to gain deeper understanding not only the behavior of solutions but also that of its derivatives. Also there is a need to study qualitative property, like stabilty of the system for $r = 3$ in the equation. At this critical value of the exponent, it is possible to make use of scale-invariant property of the equation of BFE which has not been addressed by previous researchers. We study the BF equation with initial data in $\dot{H}^s$ and $f(u)$ satisfying different properties. Existence results of global solution are obtained when $f(u)$ satisfies Lipschitz continuity condition, when $f(u) = \beta|u|^{r-1}u$ for $r > 1$ where $\beta > 0$ is a Forchheimer constant. Global existence of weak solution is also obtained in a critical space $\dot{H}^{\frac{1}{2}}$ for critical value of the exponent $r = 3$. Also for the case $f(u) = \beta\nabla^4 u$, global existence result is obtained for BFE for $f(u)$ satisfying the biharmonic function for initial data in L^2. Stability of global weak solution is investigated in a critical homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}$ using Profile Decomposition and we answer the question: If $B_{BF}^{L^3}$ is a ball in L^3 with center zero and if the members of $\dot{H}^{\frac{1}{2}} \cap B_{BF}^{L^3}$ generate global solutions of BFE, is it possible to obtain an a priori estimate for those solutions given the sequences of solutions in $\mathbb{R}^3$ associated with sequences of initial data bounded in $\dot{H}^{\frac{1}{2}}$? in affirmative. Regardless of initial data, $u(t, x)$ and corresponding pressure $p(t, x)$ become instantly smooth until possibly a singularity occurs. However, if one accepts smallness restriction of initial data, solutions remain smooth globally in time. We define

$$\rho_{max} = sup\ \{\rho > 0\ :\ for\ ||\varphi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \rho\} \tag{1.4.2}$$

and corresponding $u(t, x)$ is a global strong solution of (BFE)

Global existence of solutions would imply $\rho = +\infty$, and we assume that singularity in (BFE) is possible i.e $\rho < +\infty$. We show that the reason ρ_{max} could be finite is when finite-time singularities appear in the solution $u(t, x)$ for some initial data $\varphi(x)$.

1.5 Motivation of the study

The motivation of this work is to study the qualitative properties of solutions of Brinkman-Forchheimer equations (BFE) in a homogeneous Sobolev space by exploring the scale-invariant property of the equation.

$$(BFE)\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + f(u) &= 0 \quad \text{in } \mathbb{R}^+ \times R^3 \\ \nabla\cdot u &= 0 \\ u\,|_{t=0} &= u_0 \end{cases} \tag{1.5.1}$$

The scale-invariant property of the equation entails the following:

Given any real number λ,

$$u = BF(u_0) \text{ if and only if } u_\lambda = BF(u_{0,\lambda})$$

and u_λ is the rescaling of the velocity field u:

$$u_\lambda(t,x) = \lambda u(\lambda x, \lambda^2 t) \text{ for } \lambda > 0$$

and p_λ is the rescaling of the pressure function p:

$$p_\lambda(t,x) = \lambda^2 p(\lambda x, \lambda^2 t) \text{ for } \lambda > 0$$

and

$$u_{0,\lambda}(x) = \lambda u_0(\lambda x)$$

and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ norm is conserved under the transformation $u_0 \to u_{0,\lambda}$

1.6 Knowledge gap

This work focuses on existence results and stability of Brinkman-Forchheimer equations with respect to initial data in a homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$. The main interest of researchers in this field so far has been restricted to study the structural stability of the equation in some narrow sense. This concerns continuous dependence of solution of Brinkman-Forchheimer equation on its coefficients in L^2. Other interests concern the study of the large time behavior of solutions as well as existence of global attractor to BFE for very restrictive values and ranges of the exponent r in the equation. Previous works in the literature only focused on structural stability, existence of global attractors and some other properties of solution in L^2 and other functional spaces. Analysis of solutions of BFE in some generalized forms defined on critical Sobolev spaces has not been widely explored in the literature and this is a major motivation for this work.

1.7 Research questions

The research questions are as follows::

(1) If the initial data u_0 is in homogeneous Sobolev space $\dot{H}^s$, can we obtain existence results of solution of BFE associated with the initial conditions when the damping term $f(u)$ is continuous, continuously differentiable and satisfies Lipschitz condition?

(2) If for some open ball B^{L^3} in L^3 with center zero and the elements of $\dot{H}^{\frac{1}{2}} \cap B^{L^3}$ generate global solutions of BFE, can a priori estimates be obtained for those solutions given the sequences of solutions in $\mathbb{R}^3$ associated with bounded sequences of initial data in $\dot{H}^{\frac{1}{2}}$?

(3) If $\rho_{max} < \infty$ can there be an initial datum $\varphi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with

$$||\varphi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \rho_{max}$$

such that the solution $u(t, x)$ of BFE develops a singularity in finite time?

It is shown that the answers to these questions are positive and brought about by some auxiliary results developed in the course of the work.

Chapter 2
LITERATURE REVIEW

2.0 Introduction

Fluid flow and transport processes through porous media is an area of interest in technical as well as various scientific fields. The study of porous media flow has applications in petroleum engineering, ground water hydrology as well as agriculture irrigation chemical reactors and drainage. Research efforts have been directed by many scientists towards the way to develop a deep understanding on the transport process and the flow by experiment and numerical analysis. Brinkman-Forchheimer equations play a very important role in the study of fluid flows in a porous medium. The equations are also extremely useful in the study of non-Newtonian fluid mechanics. This is a vast subject that has gained much attention in biology, chemical engineering and geophysics. Non-Newtonian fluid phenomena is one of the deviating situations from Darcy's law in nature. A lot of work has been done towards a better understanding of the equations. Although numerical study of the equations is very scarce in the literature, much has been done on the analytical approach in different functional settings. The major focus of researchers in this field has been on the existence of solution as well as existence of global attractor. Some qualitative properties of solution like structural stability are also given much attention. Structural stability is a type of stability that reflects the effect of small changes in coefficients of the equation on the solutions.

The study of BFE in the critical case is a very rich and interesting one which takes into account and makes use of the special structure and scale-invariant property of the equation for the critical value of the exponent. Researchers in this area of study have not widely explored scale-invariant property of the equation to study its qualitative properties of solution. This work focuses on the existence results considering different conditions imposed on the damping term $f(u)$. The three conditions are considered separately for the existence results. We consider continuity condition, Lipschitz condition and differentiability on the damping term. Stability results and other qualitative properties of the solutions of the equations are also obtained in homogeneous Sobolev spaces

$$(BFE)\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + f(u) = 0 & \text{in } [0,T]\times\mathbb{R}^3 \\ \nabla\cdot u = 0 & \text{in } \mathbb{R}^3 \\ u\,|_{t=0} = u_0 & \text{in } \mathbb{R}^3 \end{cases} \tag{2.0.1}$$

where $u(t;x) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ is the velocity, $p(t;x) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ is the associated pressure and $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is given divergence-free initial data, the constant ν is the positive Brinkman coefficient (effective viscosity). The divergence free condition is for the fluid incompressibility.

2.1 Existing work

Many researchers used the Forchheimer law to describe high velocity flow. They supported their choice by stating that the Forchheimer law is responsible for the inertial effects of high velocity.

The Darcy-Forchheimer law is what BF model is based on. It was originally derived by (C. Hsu and P. Cheng, 1990) in its classical configuration ($r = 2$, $\alpha > 0$, $\beta > 0$) in thermal dispersion settings in porous media by using the volume averaging method of the temperature and velocity deviations in pores. (Nield, 1991) discussed the formulation, validity as well as the limitation of the BFE.

Some researchers have worked on continuous dependence of solutions of BF equation on coefficients in the equation and the convergence as Brinkman coefficient, $\nu \to 0$. Continuous dependence on coefficients of equations of solutions can be seen as a kind of structural stability that reflects the effect of small changes in coefficients of the models on the solutions. (Payne and Straughan, 1999) considered the initial-boundary value problem of Brinkman-Forchheimer equation with $r = 2$ that describes the flow of fluid in porous media. It was shown that the solutions of the medel depends continuously on the coefficients β and ν in L^2 norm. Celebi et al. (2005) showed continuous dependence on these coefficients in a stronger norm, that is, in H^1 norm and they used the function spaces

$$H_0^1(\Omega, R^3) = \{u \in H_0^1(\Omega, R^3) : \nabla \cdot u = 0\}$$

and

$$L^2(\Omega, R^3)$$

and the latter space denotes the closure of $H_0^1(\Omega, R^3)$ in $L^2(\Omega, R^3)$. They proved that the solution of the BF depends continuously on coefficient β in $H^1(\Omega)$ norm. Faedo-Galerkin method was used to obtain the existence and uniqueness theorem. (Liu & Lin, 2007) also worked on structural stability for BFE and the main tools used were Cauchy-Schwarz as well as Arithmetic Geometric Mean Inequalities. These authors assumed that Ω is a domain which is simply connected and is bounded with boundary $\partial\Omega$ in $\mathbb{R}^3$. The boundary condition was imposed

$$u_i = 0 \text{ on } \quad \partial\Omega \times \{t > 0\}$$

as well as initial condition

$$u_i(x, 0) = f_i(x).$$

They studied continuous dependence for Brinkman and Forchheimer coefficients for different values of the coefficients and convergence as Forchheimer coefficient tends to zero. Peter et al. (2015) introduced a continuous data assimilation algorithm for a 3D BFeD model of porous media with boundary conditions which is periodic with period L, and the only basic periodic domain is given by $\Omega = [0, L]^3$ or Dirichlet boundary conditions with no-slip

$$u|_{\partial\Omega} = 0,$$

and $\partial\Omega$ is the boundary of a domain Ω which is smooth. The extra term $\gamma|u|^{2m}u$ was introduced for the pumping, when $\gamma < 0$, while $\beta|u|^{r-1}u$ is for modeling of the damping when $\beta > 0$ and $r > 1$, $m \geq 0$. At the mathematical level, the method of continuous data assimilation was introduced first by (Azouani,2014) for 2D NSE. A follow-up work was done by Albanez et al. (2015) and Farhat et al. (2016). Louaked et al. (2016) used a pseudocompressibility technique for the equation, (Payne and Straughan, 1999) worked on continuous dependence and convergence for the BFE. (Djoko and Razafimandimby, 2017) studied the BFE by regularization with the Faedo-Galerking approach under slip boundary conditions and they also discussed the continuity of the solutions with respect to coefficients in the BFE and they showed stability of the weak solution of the stationary problem. .

The behavior of solutions of BFE at large time as well as global attractor existence has been researched for very restrictive values and ranges of the parameter r. The solutions large-time behavior and the corresponding infinite dimensional dynamical systems are essential aspects in recent times in the field of nonlinear evolutionary systems. For instance; the inertial manifold, existence of global attractors, pullback attractors, uniform attractors and their fractal dimensions for the BFE with the unique solution have been studied by many researchers since 1980s. Varga et al. (2011) proved existence of dissipative solutions which is regular and the global attractors for BFE with polynomial growth rate nonlinearity. To obtain the result, they proved the maximal estimate which is regular for the associated semi-linear stationary Stokes problem by employing modified nonlinear localization method. The applications of their results to the BFE with the NSE inertial term were also given considerations. (Xue-Li Song and Yan-Ren Hou, 2011) showed that the strong solution of the BFE has global attractors in V and $H^2(\Omega)$ with $u_0 \in V$ and $\Omega \in R^3$ is bounded for $\beta > 0$ and $\frac{7}{2} \leq r \leq 5$ and

$$H = cl_{(L^2(\Omega))^3} T,$$
$$V = cl_{(H_0^1(\Omega))^3} T,$$

where

$$T = \{u \in (C_0^\infty(\Omega))^3 : \text{div} u = 0\}$$

and cl_X is the closure in the space X. (Xue and Yan, 2012) worked on the non-autonomous BFE. By Galerkin approximation method, they gave the existence as well as the uniqueness of weak solutions for the model and investigated the weak solution asymptotic behavior, the existence as well as structures of the (H, H)-uniform attractor and (H, V) -uniform attractor. Then they proved that L^2 and H^1 -uniform attractors are the same. Lingrui et al. (2016) considered the large-time behavior taking for instance existence of attractors for autonomous as well as nonautonomous BFE. By decomposition method they circumvent the difficulties for the existence of absorbing sets and semigroup asymptotical compactness to prove the attractors generated by a global solution for the autonomous BFE.

Another area of interest to researchers is on existence and uniqueness of solutions of BFE. (Cai and Jiu, 2008), by utilizing Galerkin approximation, they showed that BFE has global weak solutions in case $r \geq 1$. The authors, using Fourier transform, worked on the compactness of approximated solutions. By Gagliardo-Nirenberg inequality, they also obtained the global strong solutions when $r \geq \frac{7}{2}$ and uniqueness result in case $\frac{7}{2} \leq r \leq 5$. Zujin et al. (2010) improved on these results and they showed that the Cauchy problem of the BFE has global strong solution for any $r > 3$ and uniqueness when $3 < r \leq 5$. Hakima et al. (2016) proved global existence and uniqueness of solutions of anisotropic BF equation without any assumption that the initial data is small . This improves on the result for anisotropic NSE. They showed that the a smoothing effect is as a result of the damping term $\beta|u|^{r-1}u$ in vertical velocity. (Markowich and Trabelsi, 2016) showed existence as well as uniqueness of solutions for different values of r. Their work is based on the estimate of maximal regularity for the associated stationary Stokes problem by employing nonlinear localization technique to prove. (Poitr and Dongming, 2017) used BFeDE to model some porous medium flow in chemical reactors of packed bed. The results concerning the existence as well as the uniqueness of a weak solution are presented for nonlinear convective flows in media with variable porosity and for small data. Furthermore, the Finite Element Approximation to the flow profiles in a fixed bed reactor were presented for several Renolds numbers at the non- Darcy range. Also, Poitr (2017) presented the Local Projection Stabilization (LPS) for the linearized BFeDE with high Renolds number. The equation was used to model porous medium flows in chemical reactor of parked bed type. Finite element analysis was presented for the case of non-constant porosity. The optimal error bounds for the velocity and pressure errors were justified numerically. (Karol and Robinson, 2017) obtained global smooth solutions in time for the convective BFE on a periodic domain, absorption exponent $r > 3$. Also for $r = 3$, it was proved that global solutions exist which also regular with the relation $4\mu\beta \geq 1$ satisfied by the coefficients. Additionally, they showed that for the exponent $r = 3$ every weak solution verifies the energy equality which is continuous into space L^2. The existence of a strong global attractor was also obtained by invoking evolutionary systems theory.

(Escauriaza and Mitrea, 2004) gave an analysis on Lipschitz domains of the transmission problems in R^n of the Laplace operator using a layer potential technique. (Mitrea and Wright, 2012) employed in their analysis of bvps the integral layer potentials for the Stokes system in Lipschitz domains in R^n $(n \geq 2)$. Kohr et al. (2013) described the concept of pseudodifferential Brinkman operator as a differentiable matrix type operator on compact Riemannian manifolds with variable coefficients. (Dindos and Mitrea, 2004) employed the integral layer potentials for the NSE to prove well-posedness in Besov and Sobolev spaces of Poisson problems with Dirichlet condition on C^1 and Lipschitz domains are considered in compact Riemannian manifolds.

Kohr et al. (2013) obtained the existence result by combining both the fixed point theorem and the integral layer potentials of the Stokes and Brinkman systems for a nonlinear Neumann-transmission problem with data obtained in L^p, Besov and Sobolev spaces.

Mirela et al. (2016) obtained existence as well as uniqueness results in L^2-weighted Sobolev spaces for transmission problems in the three dimensional Euclidean space for the Stokes as well as Darcy-Forchheimer-Brinkman systems in two complementary Lipschitz domains i.e both bounded Lipschitz domain with connected boundary and complementary Lipschitz domain $\mathbb{R}^3 \setminus \bar{\Omega}$. A layer potential method was explored for both the Brinkman and the Stokes systems combining it with a fixed point theorem in order to prove the results for small initial data in L^2- based Sobolev spaces.

(Jha and Kaurangini, 2011) presented a approximate analytical solution in parallel-plates for steady flow channels filled with porous materials modeled by nonlinear BFE with extended Darcy model. The comparison of the results were made with the ones obtained from implicit finite-difference solutions of the corresponding flow problem which is time-dependent. It was observed that the flow solution which is time dependen yielded the same steady state values gotten by employing the approximate analytical method.

Frittz et al. (2019) presented a mathematical analysis of both local and nonlocal phase-field of tumor growth that incorporates time-dependent Darcy-BFE of convective velocity fields and long-range cell interactions. They provided existence analysis which is complete.They used continuum mixture theory balance laws and mesoscale versions of the Ginzburg-Landau energy. Additionally, analysis of a parameter sensitivity was described and quantified. They employed statitical variances of model output and notion of active subspaces. They found out that the two approaches yielded a similar conclusions on the sensitivity of some quantities considered.Numerical experiment results were also obtained by finite element discretization of the model

In this thesis, we use Galerkin approximation and Profile decomposition in the context of Brinkman-Forchheimer equations. To our knowledge, the study of BFE using profile decomposition method for a critical value of exponent 'r' remains an open problem. The theory of profiles was introduced by (Gerard, 1998) for the description of the defect of compactness in the Sobolev embeddings and it is used by (Bahouri & Gerard, 1999) to investigate subtil properties of the quintic wave equation on $\mathbb{R}^3$. Then S. Keraani used it for studying some semilinear Scrodinger equations and it turns out to be an important tool in the works of C. Kenig and F. Merle about blow up in semilinear critical Schrodinger equations. The theory of profiles has been used for the incompressible NSE by Galagher (2001) and (Rusin and Sverak, 2011). Let us also mention that the profiles has been used by (Kenig and Koch, 2011) and by (Koch and Planchon, 2013) to revisit and extend the blow up result of Iskauriaza et al. (2003).

The study of compactness in Sobolev embeddings was pioneered by Lions (1985), Tarta (1990) and Gerard (1991). Our source of inspiration in this thesis is the work of Gerard (1998) in which the defect of compactness of the critical Sobolev embedding $H^s \subset L^p$ is explained in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p

In the pioneering works of (Bahouri and Gerard, 1999) (for the critical 3D wave equation) and (Merle and Vega, 1998) (for the critical 2D Schrodinger equation(SE)), this type of decomposition was introduced in the study of nonlinear pdes. (Bahouri and Gerard, 1999) ideas were revisited by Keraani (2001) and Gallagher et al. (2013) in the setting of the SE and NSE respectively, with the aim to give a description to the concept of the structure of sequences (which is bounded) of solutions to those equations. These techniques have since then been employed and used successfully when considering the study involving blow-up of solutions to nonlinear pdes in various settings; for instance (Hmidi and Keraani, 2005) worked on blowup theory for the critical nonlinear Schrodinger equations, (Kenig and Merle, 2008) worked on global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation, W. Rusin and V. Sverak worked on minimal initial data for potential singularities for NSE, (Jia and Sverak, 2013) worked on minimal L^3-initial data for potential Navier-Stokes singularities, (Gallagher, koch & Planchon, 2013) considered a profile decomposition method to the $L_t^\infty(L_x^3)$ NSE regularity criterion.

Chapter 3
METHODOLOGY

3.1 Introduction

This chapter discusses methods employed in this work. The various inequalities used to obtain bounds in the proofs of our results as well as some basic results are discussed in section two. Section three deals with Profile decomposition method used to obtain stability results and other qualitative properties of solution of BFE. Section four addresses Galerkin method used to obtain existence of solution when the damping term is continuous, continuously differentiable and when it satisfies Lipschitz condition. Some useful auxiliary results relating to decomposition of data are discussed in section five. The decomposition of linear part (the heat equation) is also discussed in this section.

3.2 Some useful inequalities

Some inequalities in L^p spaces used in this work are considered in this section.

a **Holder inequality:**

For $1 \le p \le \infty$, setting

$$p' = \frac{p}{(p-1)}$$

then Holder inequality is given by

$$\int_\Omega |uv| \le ||u||_p ||v||_{p'}$$

$\forall\ u \in L^p(\Omega)$, $v \in L^{p'}(\Omega)$ (Miranda 1978).

The p' is the conjugate of p. Specifically, the inequality shows that the bilinear form $\langle u, v\rangle$ makes sense whenever $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$. When $p = 2$, the inner product and the associated norm are connected by

b **Cauchy-Schwarz inequality:**

$$|\langle u, v\rangle| = \left|\int_\Omega u(x)v(x)dx\right| \leq \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v(x)|^2 dx\right)^{\frac{1}{2}}$$

. $\forall\ u, v \in L^2$.

Generalized Holder inequality is given by

$$\int_\Omega |u_1 u_2 \cdots u_m| \leq ||u_1||_{p^1} ||u_2||_{p^2} \cdots ||u_m||_{p^m},$$

where $u_1 \in L^{p1}(\Omega)$, $u_2 \in L^{p2}(\Omega), \cdots, u_m \in L^{p^m}(\Omega)$, and

$$\sum_{i=1}^{m} p^{-1} = 1, 1 \leq p_i \leq \infty, i = 1, ..., m.$$

c **Young inequality:**

$$ab \leq \frac{\epsilon a^p}{p} + \epsilon^{\frac{-p'}{p}} \frac{b^{p'}}{p'} (a, b, \epsilon > 0) \tag{3.2.1}$$

$\forall\ p \in (1, \infty)$. When $p = 2$, it is Cauchy inequality.

d **Minkowski inequality:**

$$||u + v||_q \leq ||u||_q + ||v||_q, u, v \in L^q(\Omega) \tag{3.2.2}$$

and the

e **Interpolation inequality:**

$$||u||_q \leq ||u||_s^\theta ||u||_r^{1-\theta} \tag{3.2.3}$$

valid for all $u \in L^s(\Omega) \cap L^r(\Omega)$ with $1 \leq s \leq q \leq r \leq \infty$ and

$$q^{-1} = \theta s^{-1} + (1-\theta)r^{-1},\ \theta \in [0,1]$$

are consequences of Holder inequality.

f **Ladyzhenskzaya's inequality:**

Let $\Omega \subset \mathbb{R}^n$ be a domain for $n = 2$ or $n = 3$, and let $u : \Omega \longrightarrow \mathbb{R}$ be a weakly differentiable function on $\partial\Omega$ in the sense of trace (i.e, u is the limit in $H^1(\Omega)$ of a sequence of smooth functions with compact support). Then $\exists$ a constant C depending only on Ω in the case $n = 2$, such that

$$||u||_{L^4} \leq C||u||_{L^2}^{\frac{1}{2}}||\nabla u||_{L^2}^{\frac{1}{2}},$$

and for $n = 3$,

$$||u||_{L^4} \leq C||u||_{L^2}^{\frac{1}{4}}||\nabla u||_{L^2}^{\frac{3}{4}}$$

Both the 2D and 3D versions of Ladyzhenskzaya's inequality are special cases of the Gagliardo-Nirenberg interpolation inequality:

$$||u||_{L^p} \leq C||u||_{L^q}^{\alpha}||u||_{H_0^s}^{1-\alpha},$$

which holds whenever $p > q \geq 1$, $s > n(\frac{1}{2} - \frac{1}{p})$

Sobolev Embedding Theorems (Bruce K. Driver, 2001):

Suppose $\Omega \subset \mathbb{R}^n$ is open with C^1- boundary, $p \in [1, \infty)$, $k,\ l \in \mathbb{N}$ and $k \geq l$

1. If $p < \frac{d}{l}$ then

$$W^{k,p}(\Omega) \hookrightarrow W^{k-l,q}(\Omega)$$

provided $q = \frac{dp}{d-pl}$ i.e q solves

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n} > 0$$

and $\exists\ C < \infty \ni$

$$||u||_{W^{k-l,q}(\Omega)} \leq C||u||_{W^{k,p}(\Omega)}$$

$\forall\ u \in W^{k,p}(\Omega)$

2. If $p > \frac{n}{k}$, then $W^{k,p}(\Omega) \hookrightarrow C^{k-\frac{n}{p}}(\Omega)$ and $\exists\ C < \infty \ni$

$$||u||_{C^{k-\frac{n}{p}}(\Omega)} \leq ||u||_{W^{k,p}(\Omega)}$$

$\forall\ u \in W^{k,p}$

g **Gronwall's Inequality:**

Let $u(t) \geq 0$ and $\varphi(t) \geq 0$ be continuous non-negative real-valued functions defined on the interval $0 \leq t \leq T$ and $u_0 \geq 0$ is a constant with a nonnegative value. If u satisfies

$$u(t) \leq u_0 + \int_0^t \varphi(s)u(s)ds \qquad t \in [0,t]$$

then

$$u(t) \leq u_o exp(\int_0^t \varphi(s)ds) \qquad t \in [0,t]$$

In particular, if $u_0 = 0$ then $u(t) = 0$

Remark 3.2.1

Because $p < \frac{n}{l}$. We have the following bounded inclusion maps

$$W^{k,p}(\Omega) \hookrightarrow W^{k-1,p_1}(\Omega) \hookrightarrow W^{k-2,p_2}(\Omega)... \hookrightarrow W^{k-l,p_l}(\Omega).$$

- **Gagliardo-Nirenberg-Sobolev inequality**:

$$||u||_{L^q} \leq C||\nabla u||_{L^p(\mathbb{R}^d)}\ \forall\ u \in C_c^1(\mathbb{R}^d).$$

For $\lambda > 0$, let $u_\lambda(x) = u(\lambda x)$. Then

$$||u_\lambda||_{L^q}^q = \int_{\mathbb{R}^d} |u(\lambda x)|^q dx = \int_{\mathbb{R}^d} |u(y)|^q \frac{dy}{\lambda^d}$$

and hence

$$||u_\lambda||_{L^q} = \lambda^{-\frac{d}{q}} ||u||_{L^q}.$$

Moreover,

$$\nabla u_\lambda(x) = \lambda(\nabla u)(\lambda x)$$

and hence

$$||\nabla u_\lambda||_{L^p} = \lambda||(\nabla u)\lambda||_{L^p} = \lambda\lambda^{-\frac{d}{p}}||\nabla u||_{L^p}$$

Remark 3.2.2

If the inequality is to hold $\forall\ u \in C_c^1(\mathbb{R}^d)$ we must have

$$\lambda^{-\frac{d}{q}}||u||_{L^q} \ = \ ||u_\lambda||_{L^q} \ \leq \ C||\nabla u_\lambda||_{L^p(\mathbb{R}^d)} \ = \ C\lambda^{\frac{1-d}{p}}||\nabla u||_{L^p} \quad \forall\lambda \ > \ 0$$

and its possibility depends on if

$$\frac{1-d}{p} + \frac{d}{q} = 0$$

i.e $\frac{1}{p} = \frac{1}{d} = \frac{1}{d} + \frac{1}{q}$

For $p \in [1, d]$, let $p^* = \frac{dp}{d-p}$ with convection that $p^* = \infty$ if $p = d$. i.e $p^* = q$.

Lemma 3.2.3(Bruce K. Driver)

Suppose $K_m : X \to Y$ are compact operators and $||K - K_m||_{L(X,Y)} \to 0$ as $n \to \infty$ then K is compact.

Theorem 3.2.4 (Bruce K. Driver, 2001):

Let $0 < p < 1$ and so $p' = \frac{p}{(p-1)} < 0$. Suppose $f \in L^p(\Omega)$ and

$$0 < \int_\Omega |g(x)|^{p'} dx < \infty.$$

Hence

$$\int_\Omega |f(x)g(x)|dx \leq \{\int_\Omega |f(x)|^p dx\}^{\frac{1}{p}} \{\int_\Omega |g(x)|^{p'} dx\}^{\frac{1}{p'}}$$

Remark 3.2.5

Assume $fg \in L^1(\Omega)$; if not, the lhs of the inequality above is infinite. Set $\phi = |g|^{-p}$ and $\psi = |fg|^p$ so $\phi\psi = |f|^p$. Then $\psi \in L^q(\Omega)$ for $q = \frac{1}{p} > 1$, and because $q' = -pq'$ where $q' = \frac{q}{(q-1)}$ and we obtain $\phi \in L^{q'}(\Omega)$. By Holder inequality, we obtain

$$\int_\Omega |f(x)|^p dx = \int_\Omega \phi(x)\psi(x)dx \leq ||\psi||_q ||\phi||_{q'}$$

$$= \{\int_\Omega |f(x)g(x)|dx\}^p \{\int_\Omega |g(x)|^{p'} dx\}^{1-p}$$

Theorem 3.3.6 (The Hausdorff-Young inequality):

If $1 \leq p \leq 2$ we have $||\hat{f}||_{L_{p'}} \leq (2\pi)^{\frac{n}{p'}} ||f||_{L^p}$

- **(Triangle Inequality):**

Let $m, n \in L^p(\Omega)$. Then $m + n \in L^p(\Omega)$ and $||m + n|| \leq ||m||_p + ||n||_q$

The next theorem (statement only) is also classical and addresses the density in Sobolev spaces

Theorem 3.2.7 (Density)[Michael Renardy, 2001]:

The space $C^\infty(\overline{\mathbb{R}^n_+}) \cap H^k(\mathbb{R}^n_+)$ is dense in $H^k(\mathbb{R}^n_+)$ $\forall\ k \in \mathbb{N}$ and $C^\infty(\overline{\mathbb{R}^n_+})$ is the collection of all smooth functions that $k-th$ derivative is continuous up to

$$\partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1} \times \{x_n = 0\}$$

Lemma 3.2.8 (Bruce K. Driver, 2001) :

Suppose $j \in \mathbb{N}$ and $j > \frac{n}{p}$ $(j \geq 1; \text{if } p > d\ j \geq 2\ p = n)$, then

$$W^{j,p}(\Omega) \hookrightarrow C||u||_{W^{j,p}(\Omega)}$$

and there exists C such that

$$||u||_{C^{j-(\frac{n}{p})}(\Omega)} \leq C||u||_{W^{j,p}(\Omega)}$$

It is sufficient to show that estimate holds $\forall\ u \in C^{\infty}(\overline{\Omega})$

$$\text{For} p > n \text{and } |\alpha| \leq j-1,$$

$$||\partial^{\alpha} u||_{C^{0,j-(\frac{n}{p})}(\Omega)} \leq C||u||_{W^{1,p}(\Omega)} \leq C||u||_{W^{j,p}(\Omega)}$$

The following theorem (statement only) gives a brief summary of embedding theorems involving Sobolev spaces

Theorem 3.2.9 (Bruce K. Driver, 2001):
Let $p \in [1, \infty]$ and $u \in W^{1,p}(\mathbb{R}^n)$. Then

1. **Morrey's Inequality.** If $p > n$, then $W^{1,p} \hookrightarrow C^{0,1-\frac{n}{p}}$ and

$$||u^*||_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(\mathbb{R}^n)}$$

2. If $p = n$ there is a space like L^{∞} which is $BMO \ni$

$$W^{1,p} \hookrightarrow BMO$$

3. **GNS Inequality:** If $1 \leq p < n$, then $W^{1,p} \hookrightarrow L^{p^*}$

$$||u||_{L^{p^*}} \leq n^{-\frac{1}{2}} \frac{p(n-1)}{n-p} ||\nabla u||_{L^p}$$

where

$$p^* = \frac{np}{n-p}$$

or

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

3.3 Profile decomposition method

The theory of profiles was pioneered by P. Gerard in 1998 in order to give a description of the compactness defect of the Sobolev embeddings. It was used by H. Bahhouri and P. Gerard (1999) (for the critical 3D wave equation) to investigate subtil properties of the quintic wave equation on $\mathbb{R}^3$ and F. Merle and L. Vega (1998) (for the critical 2D Schrodinger equation). The ideas of Bahhouri H. and Gerard P. (1999) were revisited by Keraami S. (2001) and Gallagher I., Iftimie D. and Planchon F.(2013) in the settings of the Schrodinger and NSE with the aim to give an explanation of sequences of solutions to those equations.

The first basic concept to consider in the theory of profiles from the point of view of Brinkman-Forchheimer equation is the concept of scales and cores.

Definition 3.3.1 A sequence $(h_n^j, x_n^j)_{(j,n)\in\mathbb{N}^2}$ of $[0,\infty] \times \mathbb{R}^3$ is a sequence of scales as well as cores if $j \neq k \Longrightarrow$

$$\begin{cases} \text{either } \lim_{n\to\infty}\left(\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j}\right) & = +\infty \text{ or} \\ h_n^j = h_n^k \text{ and } \lim_{n\to\infty}\frac{|x_n^j - x_n^k|}{h_n^j} & = +\infty \end{cases} \tag{3.3.1}$$

Moreover, we denote $h_n^0 = 1$ and $x_n^0 = 0$ **Definition 3.3.2: (Isabelle Gallagher, 2001)**

Suppose that $A \subset S'(\mathbb{R}^3)$ is a Banach space $\ni$ the embedding

$$\dot{H}^s(\mathbb{R}^3) \subset A$$

is continuous. Then the following properties hold if and only if A is admissible

(i) The $||\cdot||_A$ is invariant for transformations:

$\varphi \longrightarrow \lambda\varphi(\lambda.)$ and $\varphi \longrightarrow \lambda\varphi(\cdot - \lambda)$, $\forall\lambda \in \mathbb{R}$

(ii) There exists c^A, a constant which depends on A such that if φ is a member of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $||\varphi||_A$ is smaller than c^A, then φ is in $I\!D_\infty$.

Suppose that the embedding of $\dot{H}^s(\mathbb{R}^3)$ into L^p is continuous and compact. If (φ_n) is a bounded sequence of functions in $\dot{H}^s(\mathbb{R}^3)$, converging weakly to zero in $\dot{H}^s(\mathbb{R}^3)$, then it converges strongly to zero in $L^p(\mathbb{R}^3)$; as a result, for large n , the function φ_n becomes small enough to have a global solution of (BF) in E_∞

based on point (ii) of Definition 3.3.2 if the embedding of $\dot{H}^s(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ is compact, then it is possible to associate any bounded sequence of divergence free vector fields in $\dot{H}^s(\mathbb{R}^3)$, converging weakly to zero in $\dot{H}^s(\mathbb{R}^3)$ with a sequence of global solutions of (BF) in E_∞. This leads to the problem of defects of compactness of the embedding of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$, which was ve studied by Gerard in 1998. The Theorem holds for the embedding of $\dot{H}^s(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ with $s = d(\frac{1}{2} - \frac{1}{p})$ generally

THEOREM 3.3.2 (Gerard, 1998): Let (φ_n) be a bounded sequence of functions in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then up to the subsequence extracted, it can be decomposed as follows:

$$\forall\ \ell \in \mathbb{N}\setminus\ \{0\}, \varphi_n(x) = \varphi^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j}\varphi^j\left(\frac{x - x_n^j}{h_n^j}\right) + \psi_n^\ell(x) \tag{3.3.2}$$

where the functions φ^j are in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ $\forall$ $j \in \mathbb{N}$, where (ψ_n^ℓ) is a bounded sequence in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ uniformly in $\ell \in \mathbb{N}\setminus\ \{0\}$, and satisfies

$$\lim_{\ell\to\infty}(\lim_{n\to\infty} sup||\psi_n^\ell||_{L^3(\mathbb{R}^3)}) = 0, \tag{3.3.3}$$

and where for any $j \in \mathbb{N}\setminus\ \{0\}$ and (h_n^j, x_n^j), a sequence in $(\mathbb{R}^+\setminus\{0\} \times \mathbb{R}^3)^{\mathbb{N}}$ having orthogonality property (1.4.1) $\forall$ integers $(j,k) \ni j \neq k$ for $\ell \in$

$\mathbb{N}\setminus\{0\}$,

$$||\varphi_n||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \sum_{j=0}^{\ell} ||\varphi^j||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + ||\psi_n^\ell||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + o(1), \tag{3.3.4}$$

as n goes to infinity.

Remark 3.3.3

• The Theorem holds more generally for the embedding of $H^s(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ with $s = d(\frac{1}{2} - \frac{1}{p})$.

• If φ_n is divergence free, then so it is for φ, φ^j and ψ_n^ℓ, $\forall$ integers j, ℓ and n.

• The h_n^j are the scales of φ_n, the points x_n^j are the cores of concentration, and the functions

$$\varphi_n^j(x) = \frac{1}{h_n^j}\varphi^j(\frac{x - x_n^j}{h_n^j}) \tag{3.3.5}$$

are the associated profiles

• It is noted that up to the functions φ^j rescaled, it can be assumed that for every $j \in \mathbb{N}\setminus\{0\}$, either $h_n^j = 1$ and $\lim_{n\to\infty} |x_n^j| = +\infty$, or $\lim_{n\to\infty} h_n^j$ is obtained in $\{0, \infty\}$.

• The following are noted for notation simplification

$$h_n^0 = 1,\ x_n^0 = 0,\ \text{and}\ \varphi_n^0(x) = \varphi^0(x) \tag{3.3.6}$$

• The weak limit point of $h_n^j\varphi_n(x_n^j + h_n^j)$ is the function φ^j for $j \in \mathbb{N}$

Following Definition 3.3.2., for the space A we shall define the constant

$$C_{BF}^A \in \mathbb{R}^+ \cup \{+\infty\}$$

by

$$C_{BF}^A = sup\{\rho > 0 : \bar{B}_\rho^A \cap \dot{H}^s(\mathbb{R}^3) \subset I\!D_\infty\} \tag{3.3.7}$$

where

$$B_\rho^A = \{\varphi \in A; ||\varphi||_A < \rho\}$$

$$B_{BF}^{A} = B_{C_{BF}}^{A} \tag{3.3.8}$$

Putting differently, the set B_{BF}^{A} is a ball in the space A and its intersection with $\dot{H}^s(\mathbb{R}^3)$ is a subset of $I\!D_\infty$.

3.4 Galerkin method

Galerkin method was invented by a Russian mathematician, Boris Grigoryevich Galerkin. The idea of approximating infinite-dimensional by finite-dimensional problems is known as Galerkin method. It is well known as device for doing numerical calculations by converting a continuous operator problem (such as ode or pde) to a discrete problem. It is equally useful as a theoretical tool (as we use here in this thesis). The following steps are taken to show existence of weak solutions to a particular pde using Galerkin approximations.

a **Galerkin approximations**

A weak solution is built of a pde say

$$\begin{cases} u_t + Lu &= f \quad \text{in } \Gamma_T \\ u &= 0 \quad \text{on } \partial\Gamma \times [0,T] \\ u &= g \quad \text{on } \Gamma \times \{t = 0\} \end{cases} \tag{3.4.1}$$

where solutions of finite-dimensional approximations to (3.4.1) is first constructed which then pass to limits. It is assumed that the functions $w_k = w_k(x)(k = 1, ...)$ are smooth. $\{w_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H_0^1(U)$ and orthonormal basis of $L^2(U)$. A positive integer m is fixed. We look for a function $u_m : [0,T] \to H_0^1(U)$ of the form

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k \tag{3.4.2}$$

where

$$d_m^k(t)(0 \leq t \leq T, k = 1, ..., m)$$

So that

$$d_m^k(0) = (g, w_k)(k = 1, .., m) \tag{3.4.3}$$

and

$$(u_m', w_k) + B[u_m, w_k; t] = (f, w_k)(0 \leq t \leq T, k = 1, .., m) \tag{3.4.4}$$

A function u_m of the form (3.4.2) is sought that satisfies (3.4.4) of the problem projected onto the finite dimensional subspace which is spanned by $\{w_k\}_{k=1}^m$

For each integer m there is a function u_m which is unique and is of the form (3.4.2) satisfying (3.4.3) and (3.4.4).

b **Energy estimates**

As m tends to infinity, it is shown that a subsequence of the approximate solution u_m converges to a weak solution of (3.4.1) which requires some uniform estimates given as follows

Theorem (Energy estimates)

There exists a constant C, depending on U, T and the coefficient of L such that

$$max_{0 \leq t \leq T} ||u_m(t)||_{L^2(U)} + ||u_m||_{L^2(0,T;H_0^1(U))} + ||u_m'||_{L^2(0,T;H^{-1}(U))}$$
$$\leq C(||f||_{L^2(0,T;L^2(U))} + ||g||_{L^2}) \tag{3.4.5}$$

for$m = 1, 2...$

c A weak solution is then built by passing u_m to limits as $m \to \infty$.

Theorem (Browder-Minty)

Let X be a real, reflexive Banach space and let

$$T : X \longrightarrow X^*$$

be bounded continuous, coercive and monotone. Then for any $g \in X^*$, there exists a solution u of the equation

$$T(u) = g \tag{3.4.6}$$

i.e

$$T(X) = X^*$$

and X and X^* are separable.

The main focus here is to approximate equation of the form (3.3.1) by a finite dimensional problem of the form

$$T_n(u_n) = g_n \tag{3.4.7}$$

It is shown that there exists $u \in X$, $g \in X^*$ and a subsequence u_n such that $u_n \rightharpoonup u$ in X and $T(u_n) \rightharpoonup g$ in X^*. These convergences guarantee that u is a solution to the problem.

3.5 Some auxillary results

For the case when the damping term $f(u) = \beta|u|^{r-1}u$ and in the critical value of exponent $r = 3$, the homogeneous BFE and NSE have the same scaling.

$$(NSE) \begin{cases} \partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^3 \\ u \mid_{t=0} &= u_0 \quad \text{in } \mathbb{R}^3 \end{cases} \tag{3.5.1}$$

The damping term $f(u)$ is a resistance to the fluid flow. The scale-invariance property is lost given any other values of r different from 3. This can be seen from the following:

Proposition 3.5.1: (Karol, Hajduk, & Robinson, 2017)

Let $\Omega = \mathbb{R}^n$ and let u_λ be the parabolic rescaling of the velocity field u:

$$u_\lambda = \lambda u(\lambda x, \lambda^2 t) \; for \; \lambda > 0,$$

and let p_λ be the rescaling of the pressure function p:

$$p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \; for \; \lambda > 0.$$

If u and p solve the BF equations, then u_λ, p_λ satisfy

$$\partial_t u_\lambda - \nu \Delta u_\lambda + (u_\lambda \cdot \nabla) u_\lambda + \nabla p_\lambda + \lambda^{3-r} \beta |u_\lambda|^{r-1} u_\lambda = 0$$

The dilation invariance plays a paramount role in the analysis of (NSE) and classify of functional spaces into sub-/super-/critical depending on how the scaling affects the norm. It is noted that the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $L^3(\mathbb{R}^3)$ norms are conserved under the transformation $u_0 \to u_{0,\lambda}$. Also, result of H. Fujita and T. Kato for Navier-Stokes equations can be stated as follows:

Theorem 3.5.2: (Fujita and T. Kato, 1964)

Let $\phi(x) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then there exists a maximal time $T_{max}(\phi) > 0$ and a unique solution $u(t,x) = NS(\phi)$ associated with ϕ such that

$$u(t,x) \in C\left([0,T], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T], \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \; for \; all \; T < T_{max}(\phi)$$

Moreover, if

$$T_{max}(\phi) < +\infty,$$

then we have

$$\lim_{T \to T_{max}} ||u||_{L^2([0,T], \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} = +\infty$$

Also, $\exists\, C_E > 0$ such that

$$||\phi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < C_E \Longrightarrow T_{max}(\phi) = +\infty$$

and we have for any $t \geq 0$,

$$||u(t,x)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\nu \int_0^t ||u(s)||^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} ds \leq C||\phi||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \tag{3.5.2}$$

3.5.1 Decomposition of data

A sequence of divergence free vector fields (φ_n) is considered in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, and φ^0 is a weak limit point of (φ_n) in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then Theorem 3.3.2 can be applied to $(\varphi_n - \varphi^0)$, with (3.3.6) we have, ,

$$\forall \ell \in \mathbb{N}, \quad \varphi_n(x) = \sum_{j=0}^{\ell} \frac{1}{h_n^j} \varphi^j \left(\frac{x - x_n^j}{h_n^j} \right) + \psi_n^\ell(x), \tag{3.5.3}$$

where φ^j is in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for $j \in \mathbb{N}$, $\psi_n^\ell(x)$ is a bounded sequence in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and satisfies the limit (3.3.3). The sequences (h_n^j, x_n^j) are orthogonal in accordance to (3.3.1), and for every $\ell \in \mathbb{N}$, the orthogonality of the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ norms is obtained and is written in (3.3.4).

3.5.2 A profile decomposition for the heat equation

The heat equation which is the linear equation associated with BFE is considered.

$$(H) \left\{ \begin{array}{rcl} \partial_t u - \nu \Delta u & = & 0 \quad \text{in } \mathbb{R}^+ \times R^3 \\ u\,|_{t=0} & = & u_0 \end{array} \right. \tag{3.5.4}$$

Notation 3.5.3:

$H(u_0)$ denotes the solution of the heat equation (H), that can be associated with the data u_0. It is noted that if $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then $H(u_0) \in E_\infty$, and the norm E_∞

is conserved by applying H.

$u = BF(u_0) \Leftrightarrow u_\lambda = BF(u_0, \lambda)$

Proposition 3.5.4: (Isabelle Gallagher 2001)

Let (φ_n) be a collection of bounded divergence free vector fields in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, and let $\varphi^0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ be a weak limit point of (φ_n). Then up to subsequence extracted, we have

$$h_n^0 = 1,\ x_n^0 = 0,\ \varphi_n^0(x) = \varphi^0(x),$$

We define

$$u_n = H(\varphi_n) \in E_\infty$$

and

$$U^j = H(\varphi^j) \in E_\infty = C_b^0\left(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^+)\right)$$

and every integer $j \in \mathbb{N}$, $t \geq 0$ for every $x \in \mathbb{R}^3$, we can write

$$u_n(t,x) = \sum_{j=0}^{\ell} \frac{1}{h_n^j} U^j \left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right) + w_n^\ell(t,x)$$

where $w_n^j = H(\psi_n^\ell)$ is uniformly bounded in E_∞ for $\ell \in \mathbb{N}$, with

$$\lim_{\ell \to \infty} \left(\lim_{n \to \infty} sup ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \right) = 0. \tag{3.5.5}$$

Also, the sequences (h_n^j, x_n^j) are orthogonal in accordance to (3.1.10) and for every $\ell \in \mathbb{N}$

$$||u_n||_{E_\infty}^2 = \sum_{j=0}^{\ell} ||U^j||_{E_\infty}^2 + ||w_n^\ell||_{E_\infty}^2 + o(1), \tag{3.5.6}$$

when $n \longrightarrow \infty$.

Remark 3.5.5

Considering the decomposition (3.5.3) and define

$\forall j \in \mathbb{N}$, $U^j = H(\varphi^j)$ and $\forall\ (\ell, n) \in \mathbb{N}^2$, $w_n^\ell = H(\psi_n^\ell)$. By scale-invariance

$$u = H(u_0) \Leftrightarrow u_\lambda = H(u_0, \lambda)$$

with

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

and

$$u_{0,\lambda}(x) = \lambda u_0(\lambda x)$$

of (H), we have

$$u_n^j(t,x) = \frac{1}{h_n^j} U^j\left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right) = H(\varphi_n^j)$$

associated with the scale-invariance

$$\forall (j,n) \in \mathbb{N}^2,\ ||u_n^j||_{E_\infty} = ||U^j||_{E_\infty}$$

In other to give a proof of the result involving the φ_n generating a finite time singularity of solution, we will use a stability result obtained by I. Gallagher, D. Iftimie and F. Planchon (2013) in the setting of the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ data for NSE. We have the following

Theorem 3.5.6: (Gallagher I., Iftimie D. and Planchon F. [2013])
Let $u \in C(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ be a mild solution of (NSE) with initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then

$$\lim_{t \to +\infty} ||u(t,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = 0,$$

$$u(t,x) \in L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)).$$

Moreover, there exists $\epsilon = \epsilon(u)$ such that if $||u_0 - v_0||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \epsilon(u)$ then the mild solution $u(t,x)$ with initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is also global and belongs to $C(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ and

$$\begin{aligned} ||(u-v)(t,\cdot)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3} + \int_0^t ||\nabla(u-v)(s,\cdot)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} ds \\ \leq C||u_0 - v_0||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} exp\left(C\int_0^t ||v||^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} ds + \int_0^t ||v||^4_{\dot{H}^4_{\dot{H}^1(\mathbb{R}^3)}} ds\right) \end{aligned} \quad (3.5.7)$$

If a solution of Brinkman-Forchheimer satisfies the energy estimate of BFE this implies that $T_*(\varphi) = +\infty$. However, it is not clear that we cannot have $T_*(\varphi) = +\infty$ without the energy, thus we set

$$C_B = sup \left\{ \begin{array}{rcl} \rho > 0: \text{ for } ||\varphi||_{\dot{H}^{\frac{1}{2}}} < \rho, & & \\ u(t,x) & = & BF(\varphi) \text{ is a global solution of (BFE)} \end{array} \right\} \tag{3.5.8}$$

In this section, some results are presented that are used in the proofs of our theorems. Those results are classic

Proposition 3.5.7 :

There is γ, a constant such that the following statement is true.

Let $u_0 \in \dot{H}^{\frac{1}{2}}$ be a divergence-free vector field such that $||u_0||_{L^3(\mathbb{R}^3)}$ is lesser than $\gamma\nu$. Then $u_0 \in \mathbb{D}_\infty$, and if $u = BF(u_0)$, then given $t \geq 0$, we obtain

$$||u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \quad + \quad \nu \int_0^t ||\nabla u(s)||^2_{\dot{H}^{\frac{1}{2}}} ds \quad + \quad \beta \int_0^t ||u(s)||^4_{L^4} \quad \leq \quad ||u_0||^2_{\dot{H}^{\frac{1}{2}}}$$

Proof

There is a constant $c > 0$ such that if

$$||u_0||_{L^3(\mathbb{R}^3)} \leq c\nu$$

then

$$BF(u_0) \in C^0(\mathbb{R}^+, L^3(\mathbb{R}^3))$$

and we obtain

$$\forall\, t \quad ||BF(u_0)||_{L^3(\mathbb{R}^3)} \leq C||u_0|_{L^3(\mathbb{R}^3)}$$

C being a constant. More so, $L^3(\mathbb{R}^3)$ is an admissible space. Based on the definition of an admissible space, we have $\gamma \leq c$ such that if $||u_0||_{L^3(\mathbb{R}^3)} \leq \gamma\nu$ then $u_0 \in \mathbb{D}_\infty$
The estimate in the Proposition can now be proved in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for $u(t,\cdot)$ given as

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \nu||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \left|(u \cdot \nabla u(t)|u(t))_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}\right|$$

Defining

$$\wedge = \sqrt{-\Delta},$$

we obtain

$$(u \cdot \nabla u(t))|u(t)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = (u \cdot \nabla u(t)| \wedge u(t))_{L^2},$$

and by Holder inequality,

$$(u \cdot \nabla u(t)| \wedge u(t))_{L^2} \leq ||u(t)||_{L^3(\mathbb{R}^3)} \times ||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$$

Based on the

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

the embedding which is continuous, we can infer that

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \nu||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq C||u(t)||_{L^3(\mathbb{R}^3)} \times ||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$$

Choosing γ very small, we have

$$\frac{d}{dt}||u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \nu||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq 0.$$

□

The next result is giving a description on the global solution of BFE. If the initial data do not belong to $I\!D_\infty$ as defined in the next chapter, global solution of BFE associated with the intial data will fail to exist. The result is given below

Proposition 3.5.8:

Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ be a vector field, and T_{max}, $BF(u_0)$ life span. If $u_0 \notin I\!D_\infty$, then

$$T_{max} \leq \frac{1}{\nu(C_{BF}^{\dot{H}^{\frac{1}{2}}})^4}||u_0||^4_{L^2(\mathbb{R}^3)}.$$

Proof

It is a classical proof. Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ be a divergence and suppose that $u_0 \notin I\!D_\infty$

Let T_{max} be the life span of $u = BF(u_0)$. Then $\forall\ T \leq T_{max}$ and $\lambda(Y)$ stands for Lesbegue measure of Y, we obtain

$$\lambda\{t \in [0,T]; C_{BF}^{\dot{H}^{\frac{1}{2}}} \leq ||u(t,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}\} \leq \frac{1}{\nu(C_{BF}^{\dot{H}^{\frac{1}{2}}})^4}\int_0^T ||u(t,\cdot)||^4_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}dt$$

by Bienayme-Tchebytchev inequality, we obtain

$$T_{max} \leq \frac{1}{(C_{BF}^{\dot{H}^{\frac{1}{2}}})^4}||u||^2_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^3))} \times ||u||^2_{L^2(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))} \leq \frac{1}{\nu(C_{BF}^{\dot{H}^{\frac{1}{2}}})^4}||u_0||^4_{L^2(\mathbb{R}^3)}$$

3.5.3 Admissible space

In the this section, we give a definition of an admissible space. In this space, the norm of the solution of BFE is invariant at the critical value of the exponent r in the equation. This property actually enables us to apply profile decomposition to BFE. **Definition 3.5.9: (Isabelle Gallagher, 2001)** Let $A \subset S'(\mathbb{R}^3)$ be a Banach space such that the embedding $\dot{H}^s(\mathbb{R}^3) \subset A$ is continuous. Then the following properties are satisfied if and only if A is an admissible space:

(i) The $||\cdot||_A$ is an invariant norm in the transformations
$\varphi \longrightarrow \lambda\varphi(\cdot - \lambda)$, $\forall x_0 \in \mathbb{R}^3$ and $\varphi \longrightarrow \lambda\varphi(\lambda.)\ \forall\ \lambda \in \mathbb{R}$

(ii) There exists a constant c^A which depends only on ν and $A \ni$ if $\varphi \in\ \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $||\varphi||_A$ is smaller than c^A, then φ is in $\mathbb{D}_\infty$

We shall define the constant

$$C_{BF}^A \in \mathbb{R}^+ \cup \{+\infty\}$$

by

$$C_{BF}^A = sup\{\rho > 0 : \bar{B}_\rho^A \cap \dot{H}^s(\mathbb{R}^3) \subset \mathbb{D}_\infty\}, \tag{3.5.9}$$

where

$$B_\rho^A = \{\varphi \in A; ||\varphi||_A < \rho\}$$

and

$$B_{BF}^{A} = B_{C_{BF}}^{A} \tag{3.5.10}$$

The largest ball in A is given by the set B_{BF}^{A} whose intersection with $\dot{H}^s(\mathbb{R}^3)$ is a subset of $I\!D_\infty$. we have

$$C_{BF}^{A} \geq c^A$$

where c^A was defined in property (ii) of Definition 3.3.2
Suppose that the embedding of $\dot{H}^s(\mathbb{R}^3)$ into L^p is continuous and compact. If (φ_n) is a bounded sequence of functions in $\dot{H}^s(\mathbb{R}^3)$, converging weakly to zero in $\dot{H}^s(\mathbb{R}^3)$, then there will be a strong convergence to zero in $L^p(\mathbb{R}^3)$; if n is very large, the function φ_n would be very small to obtain a global solution of BFE which is unique in E_∞ following the point (ii) of Definition 3.5.9 if the embedding of $\dot{H}^s(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ is compact, then it is possible to assign with bounded sequence of divergence free vector fields in $\dot{H}^s(\mathbb{R}^3)$, converging weakly to zero in $\dot{H}^s(\mathbb{R}^3)$, a sequence of global solutions of (BF) in E_∞. This leads to defects of compactness of the embedding of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$, which was studied in 1998 by P. Gerard. It holds generally for the embedding of $\dot{H}^s(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ with

$$s = d(\frac{1}{2} - \frac{1}{p})$$

Definition 3.5.10:

The Leray-Hopf weak solution of the BFE with $u_0 \in \dot{H}^s$ is the one that satisfies the inequality

$$||u(t)||^2_{\dot{H}^s(\mathbb{R}^3)} + 2\nu \int_0^t ||u(s)||^2_{\dot{H}^{s+1}(\mathbb{R}^3)} ds + 2\beta \int_0^t ||u(s)||^{r+1}_{L^{r+1}(\mathbb{R}^3)} ds \leq C||u_0||^2_{\dot{H}^s(\mathbb{R}^3)} \tag{3.5.11}$$

for initial times $t_0 \in [0, T_{max})$, and all $t_1 \in [0, T_{max})$. For every $u_0 \in H$, $\exists$ at least one global solution of BFE

Remark 3.5.11:

It is noted that nothing will prevent the estimate the life span T_{max} with respect to some data to satisfy $T_{max} = +\infty$ with $u_0 \notin D_\infty$, in that case

$$\lim_{T\to+\infty} ||BF(u_0)||_{L^2([0,T],\dot{H}^{s+1})} = +\infty$$

Lemma 3.5.12 (Xiaojing Cai and Quansen Jiu, 2004)

Given that Y_0 and Y are Hilbert spaces that satisfy compact embedding $Y \hookrightarrow\hookrightarrow Y$. Let $0 < \alpha < 1$ and $\{v_j\}_{j=1}^{\infty} \subset L^2(\mathbb{R}; Y_0)$ that satisfies $sup_j(\int_{-\infty}^{\infty} ||v_j||_{X_0}^2 dt) < \infty$ and $sup_j(\int_{-\infty}^{\infty} |\tau|^{2\alpha}||\hat{v}_j||_X^2 dt) < \infty$. Then a subsequence of $(v_j)_{j=1}^{\infty}$ exists that strongly converges in $L^2(\mathbb{R}; X)$ to some u in the same space.

Chapter 4
RESULTS AND DISCUSSION

4.1 Introduction

This chapter discusses the results we have in this thesis. These results include: the existence of weak solutions of the BFE in homogeneous Sobolev spaces when the damping term is continuous, continuously differentiable and satisfies Lipschitz condition. Stability of the system with respect to initial data is obtained and some other qualitative properties of solution are investigated. Finite time singularities due to singularity-generating initial data are also investigated. Existence of weak solutions for the damping term satisfying Lipschitz condition is addressed in section two. Section three deals with existence of weak solutions of BFE when the damping term is biharmonic. Section four discusses the existence of weak solutions of BFE when the damping term is continuous. Section four discusses the existence of solution in the critical space when the value of the exponent in the equation $r = 3$ and $r \geq 1$. Section five centres on stability results while section six focuses on bounded energy solution of BFE and finite time singularities of solution with respect to singularity-generating initial data of BFE are discussed in section seven.

$$(BFE)\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + f(u) &= 0, \quad (t,x) \in [0,T]\times\mathbb{R}^3 \\ \nabla\cdot u &= 0 \quad (t,x)\in[0,T]\times\mathbb{R}^3 \\ u\mid_{t=0} &= u_0 \quad x\in\mathbb{R}^3 \end{cases} \tag{4.1.1}$$

where $u(t,x) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ is the velocity, $p(t,x) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ is the associated pressure and $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is the given divergence-free initial data The constant ν is the positive constant Brinkman coefficient while $f(u)$ is the damping term. The divergence free condition describes the incompressibility of the fluid. $p = p(t,x),\ x \in \mathbb{R}^3$ and $\nabla p = (\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_i},\ i = 1,2,3)$. BFE (4.1.1) describes the non-Darcy behavior of fluid flows in porous media.

4.2 Brinkman-Forchheimer equation with Lipschitz continuous damping term $f(u)$

A continuous function may not satisfy Lipschitz condition but every function that satisfies Lipschitz condition is continuous. In this section, we consider the damping term that satisfies Lipschitz condition. The BFE (4.1.1) is considered with the damping term $f(u)$ satisfying the following conditions:

(i) $f \in C(\mathbb{R}^3, \mathbb{R}^3)$ such that $f(0) = 0$ and $f(u)u \in L^{r+1}$ and $r > 1$

(ii) $f(u)$ satisfies Lipschitz condition:
$|f(u) - f(v)| \leq L(M)|u - v| \; \forall u, v \in \mathbb{R}^n$
such that $|u|, |v| \leq M$ with $L(\cdot) \in C([0, \infty))$. .

Theorem 4.2.1:

Suppose the following conditions are satisfied:

(a) u_0 is an arbitrary function in $\dot{H}^s$

(b) $f : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies (i) and (ii) above

(c) $p : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and
$\nabla p = (\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_i}, \; i = 1, 2, 3)$ exists

Then for $T > 0$, a solution $u : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ of BFE (4.1.1) exists and

$$u \in L^\infty\left([0, T]; \dot{H}^s(\mathbb{R}^3)\right) \cap L^2\left([0, T]; \dot{H}^{s+1}(\mathbb{R}^3)\right) \cap L^{r+1}\left([0, T], L^{r+1}(\mathbb{R}^3)\right)$$

and

$$sup_{0\leq t\leq T}||u||^2_{\dot{H}^s(\mathbb{R}^3)} + 2\nu \int_0^T ||\nabla u(t)||^2_{\dot{H}^s(\mathbb{R}^n)} dt + 2\beta \int_0^T ||f(u)u(t)||^{r+1}_{L^{r+1}} dt \leq ||u_0||^2_{\dot{H}^s}$$

Proof:

This is established in sequence as follows:

Step 1

We build a weak solution by first constructing solutions of certain finite-dimensional approximations to BFE (4.1.1) and then passing to limits. Since $\dot{H}^s$ is separable and C_0^∞ is dense in $\dot{H}^s$, there exists a sequence $\omega_1, \omega_2, \omega_3, ..., \omega_m$ of members of C_0^∞, in $\dot{H}^s$. Fix a positive integer m. We look for a function $u_m : [0,T] \to \dot{H}^s(\mathbb{R}^3)$ of the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i(x) \tag{4.2.1}$$

(4.2.1) is an approximate solution which satisfies the equation. By multiplying (4.1.1) by a test function $w_j \in C_0^\infty$ and integrate, we obtain the following

$$(u_m'(t), \omega_j) + \nu(\nabla u_m(t), \nabla\omega_j) + (u_m(t)\cdot\nabla u_m(t), \omega_j) + (f(u)_m, \omega_j) = 0 \tag{4.2.2}$$

$$t \in [0,T], j = 1, 2, ..., m. \text{ and } u_{0m} \longrightarrow u_0 \in \dot{H}^s,\ as\ m \longrightarrow \infty$$

Thus a function u_m is sought of the form (4.2.1) that satisfies (4.2.2) spanned by $\{w_j\}_{j=1}^m$ unto the finite dimensional subspace.

Step 2

To show that a subsequence of u_m converges to a weak solution of BFE (4.1.1), uniform estimates are needed on the approximate solutions and this follows from the following Lemma.

Lemma 4.2.2:

Let $u_0 \in \dot{H}^s$. Then given any $T > 0$, we obtain

$$sup_{0\leq t\leq T}||u_m||_{\dot{H}^s} + ||u_m||_{L^2(0,T;\dot{H}^{s+1})} + ||f_m(u)u_m||_{L^{r+1}(0,T;L^{r+1}} \leq C,$$

Proof

Multiply both sides of (4.2.2) by $g_{jm}(t)$ and summing over $j = 1, ..., m,$. By integration by parts, we obtain the following for each term

$$\begin{aligned}(u_m'(t), \omega_j)\cdot g_{jm}(t) = \sum_{j=1}^{3}\int u_m' g_{jm} w_j = \sum_{j=1}^{3}\int u_m' u_m dx = \sum_{j=1}^{3}\frac{1}{2}\int \frac{d}{dt}(u_m)^2 dx \\ \leq \frac{1}{2}\frac{d}{dt}||u_m||^2\end{aligned}$$

Similarly

$$\nu(\nabla u_m(t), \nabla\omega_j) \cdot g_{im} = \nu\sum_{j=1}^{3}\int(\nabla u_m \cdot \nabla g_{jm}w_j)dx \leq \nu||\nabla u_m||^2$$

and

$$(f(u_m), w_j) \cdot g_{im} = \sum_{j=1}^{3}\int f_m(u)u_m dx \leq ||f(u_m)u_m||_{L^{r+1}}^{r+1}$$

After getting the bound on each term, we have

$$\frac{1}{2}\frac{d}{dt}||u_m||_{\dot{H}^s}^2 + \nu||\nabla u_m||_{\dot{H}^s}^2 + \beta||f(u_m)u_m||_{L^{r+1}}^{r+1}dt \leq 0$$

using $((u\cdot\nabla)v,v)=0$. Integrate on time t over $(0,T)$, we obtain

$$sup_{0\leq t\leq T}||u_m||^2_{\dot{H}^s}+2\nu\int_0^T||\nabla u_m||^2_{\dot{H}^s}dt+2\beta\int_0^T||f(u_m)u_m||^{r+1}_{L^{r+1}}dt\leq||u_0||^2_{\dot{H}^s}$$

□

Step 3

A solution of BFE (4.1.1) is built by passing to limits as $m\to\infty$.

Invoking Lemma 4.2.2, u_m is obtained in the space $L^\infty(0,T;\dot{H}^s(\mathbb{R}^n))\cap L^2(0,T;\dot{H}^{s+1}(\mathbb{R}^n))\cap L^{r+1}(0,T;L^{r+1}(\mathbb{R}^n))$. By using Lemma 3.5.12, we prove next that u_m (or its subsequence) convergences strongly in $L^2\cap L^r([0,T]\times\mathbb{R}^3)$. $\tilde{u}_m$ is a function from $\mathbb{R}$ to $\dot{H}^{s+1}$ and on $[0,T]$, $\tilde{u}_m=u_m$, $\tilde{u}_m=0$ on $\mathbb{R}\backslash[0,T]$. In the same vein, $g_{im}(t)$ prolonged to $\mathbb{R}$. By definition, $\tilde{g}_{im}(t)=0$ for $t\in\mathbb{R}\backslash[0,T]$. The Fourier transform on variable t of $\tilde{u}_m$ and $\tilde{g}_{im}$ is given respectively by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$.

And $\tilde{u}_m$ which are the approximate solutions satisfy

$$\frac{d}{dt}(\tilde{u}_m,\omega_j)=\nu(\nabla\tilde{u}_m(t),\nabla\omega_j)+(\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t),\omega_j)+(f(u_m),\omega_j)\equiv$$
$$(\tilde{f},\omega_j)+(f(u_m),\omega_j)\ j=1,2,...,m.$$

where

$$(\tilde{f}_m,\omega_j)=\nu(\nabla\tilde{u}_m(t),\nabla\omega_j)+(\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t),\omega_j).$$

Taking Fourier transform we have

$$2\pi i\tau(\hat{\tilde{u}}_m,\omega_j)=(\hat{\tilde{f}}_m(\tau),\omega_j)+\nu(f(u_m),\omega_j))+(u_{0m},\omega_j)-(u_m(T),\omega_j)exp(-2\pi iT\tau).\tag{4.2.3}$$

where $\hat{\tilde{f}}_m$ is Fourier transforms of $\tilde{f}_m$.

Multiplying (4.2.3) by $\hat{\tilde{g}}_{jm}(\tau)$ for $j=1,...,m$ to have

$$2\pi i\tau||(\hat{\tilde{u}}_m(\tau)||^2_2=(\hat{\tilde{f}}_m(\tau),\hat{\tilde{u}}_m)+\nu(f(\hat{\tilde{u}}_m),\hat{\tilde{u}}_m))+(u_{0m},\hat{\tilde{u}}_m)-(u_m(T),\hat{\tilde{u}}_m)exp(-2\pi iT\tau).\tag{4.2.4}$$

For $v\in L^2((0,T);\dot{H}^1)\cap L^{r+1}(0,T;L^{r+1})$, we have

$$(f_m(t),v)=(\nabla u_m,\nabla v)+(u_m\cdot\nabla u_m,v)\leq C(||\nabla u_m||^2_2+||\nabla u_m||_2||v||_{\dot{H}^1}$$

For $T>0$

$$\int_0^t||f_m(t)||_{\dot{H}^{-s}}dt\leq\int_0^T C(||\nabla u_m||^2_2+||\nabla u_m(t)||_2)dt\leq C\tag{4.2.5}$$

and hence

$$sup_{\tau\in\mathbb{R}}||\hat{\tilde{f}}_m(\tau)||_{\dot{H}^{-s}}\leq\int_0^T||f_m(t)||_{\dot{H}^{-s}}\leq C\tag{4.2.6}$$

Moreover, from the assumption on $f(u)$, we have that

$$\int_0^T ||f(u)u||dt \leq C$$

which implies that

$$sup_{\tau\in\mathbb{R}}||\widehat{f(u(\tau))}|| \leq C \tag{4.2.7}$$

From Lemma 4.2.2, we have

$$||u_m(0)|| \leq C, \ ||u_m(T)|| \leq C \tag{4.2.8}$$

We deduce from (4.2.5) - (4.2.8) that

$$|\tau|||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s} \leq C(||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{s+1}} + ||\hat{\tilde{u}}_m(\tau)||_{L^p}))$$

For a fixed α, $0 < \alpha < \frac{1}{4}$, it is noted that

$$|\tau|^{2\alpha} \leq C\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}, \ \forall\tau\in\mathbb{R}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty}|\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{H^s} &\leq C\int_{-\infty}^{\infty}\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s}d\tau \\ &\leq C\int_{-\infty}^{\infty}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s}d\tau \\ &+ C\int_{-\infty}^{\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{s+1}}}{1+|\tau|^{1-2\alpha}}d\tau + C\int_{-\infty}^{\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{r+1}}{1+|\tau|^{1-2\alpha}}d\tau \end{aligned} \tag{4.2.9}$$

By Parseval equality as well as Lemma 4.2.2, the first integral on rhs of (4.2.9) is uniformly bounded on m.

By the Parseval equality, the Schwarz inequality and Lemma 4.2.2, we have

$$\int_{-\infty}^{+\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{s+1}}}{1+|\tau|^{1-2\alpha}}d\tau \leq \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}}\left(\int_0^T||u_m(t)||^2_{\dot{H}^{s+1}}dt\right)^{\frac{1}{2}} \leq C \tag{4.2.10}$$

Similarly, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{||\hat{\tilde{u}}_m(\tau)||_{r+1}}{1+|\tau|^{1-2\alpha}}d\tau &\leq \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^{\frac{r+1}{r}}}\right)^{\frac{r}{r+1}}\left(\int_{-\infty}^{+\infty}||\hat{\tilde{u}}_m(\tau)||^{r+1}_{r+1}(\tau)d\tau\right)^{\frac{1}{r+1}} \\ &\leq C\int_{-\infty}^{+\infty}(||\hat{\tilde{u}}_m(\tau)||^{\frac{r+1}{r}}_{r+1}(\tau)d\tau)^{\frac{1}{r+1}}\end{aligned} \tag{4.2.11}$$

From (4.2.9), It follows that

$$\int_{-\infty}^{+\infty}|\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s} \leq C \tag{4.2.12}$$

Hence a subsequence of u_m exists given by u_n such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;\dot{H}^s)$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(0,T;\dot{H}^{s+1})$. $\int_0^T\int_{\mathbb{R}^n}f(u)dxdt \leq C$, then $u_n \longrightarrow u$ strongly in $L^p(0,T;L^p)$ for $p<\infty$. The convergences really show that $u(x,t)$ is indeed a weak solution of the BFE. $\Box$.

The next result is a brief proof of a particular case when the damping term satisfies

Lipschitz condition. In this case the damping term $f(u) = \beta|u|^2 u$. The equation is given by:

$$(BFE_a) \begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + \beta|u|^2 u &= 0, \quad (t,x) \in [0,T]\times\mathbb{R}^3 \\ \nabla\cdot u &= 0, \quad (t,x)\in[0,T]\times\mathbb{R}^3 \\ u\mid_{t=0} &= u_0, \quad x\in\mathbb{R}^3 \\ |u| \longrightarrow 0, \quad \text{as}|x|\longrightarrow\infty \end{cases} \tag{4.2.13}$$

In this case, $f(u) = \beta|u|^2 u$ satisfies the Lipschitz condition: $|f(u) - f(v)| \leq L(M)|u-v|\ \forall\ u,v\in\mathbb{R}^3$ such that $|u|,|v|\leq M$ with $L(\cdot)\in C([0,\infty))$. For this particular case, the BFE_a (4.2.13) has the same scaling with NSE and this is crucial for the scale-invariance property of the equation in the critical homogeneous Sobolev space. The following result gives existence of weak solution to (4.2.13)

Theorem 4.2.3:

Suppose the following conditions are satisfied:

(i) u_0 is an arbitrary function in $\dot{H}^s$

(ii) $p:[0,T]\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ and
$\nabla u = (\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_i}, i=1,2,3)$ exists

Then for $T>0$, a weak solution $u:[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ of (4.2.13) exists and

$$u\in L^\infty\left([0,T];\dot{H}^s(\mathbb{R}^3)\right)\cap L^2\left([0,T];\dot{H}^{s+1}(\mathbb{R}^3)\right)\cap L^4\left([0,T],L^4(\mathbb{R}^3)\right)$$

and

$$sup_{0\leq t\leq T}||u||^2_{\dot{H}^s(\mathbb{R}^n)} + 2\nu\int_0^T ||\nabla u(t)||^2_{\dot{H}^s(\mathbb{R}^n)}dt + 2\beta\int_0^T ||u(t)||^4_{L^4}dt \leq ||u_0||^2_{\dot{H}^s}$$

Proof

This proof is established in sequence as follows:

Step 1

We build a weak solution in the sense of (1.1.11) by first constructing solutions of certain finite-dimensional approximation of (4.2.13) and the passing to limits. Since $\dot{H}^s$ is separable and C_0^∞ is dense in $\dot{H}^s$, there exists a sequence $\omega_1,\omega_2,\omega_3,...,\omega_m$ of members of C_0^∞, in $\dot{H}^s$. Fix a positive integer m. We look for a function $u_m:[0,T]\to\dot{H}^s(\mathbb{R}^3)$ of the form

$$u_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i(x) \tag{4.2.14}$$

which is an approximate solution which satisfies the equation. By multiplying the equation by a test function $w_j \in C_0^\infty$ and integrate, we obtain the following

$$(u'_m(t), \omega_j) + \nu(\nabla u_m(t), \nabla \omega_j) + (u_m(t) \cdot \nabla u_m(t), \omega_j) + (|u_m|^2 u_m, \omega_j) = 0 \quad (4.2.15)$$

$$t \in [0,T], j = 1,2,...,m. \text{ and } u_{0m} \longrightarrow u_0 \in \dot{H}^s, \ as \ m \longrightarrow \infty$$

Thus a function u_m of the form (4.2.14) is sought that satisfies (4.2.15) subspace spanned by $\{w_j\}_{j=1}^m$ unto the finite dimensional subspace.

Step 2

To show that a subsequence of the solutions u_m of the approximate problems converges to a weak solution of (4.2.13), uniform estimates are needed on the approximate solutions and this follows from the following Lemma.

Lemma 4.2.4:

Let $u_0 \in \dot{H}^s$. Then given $T > 0$, we have

$$sup_{0\le t\le T}||u_m||_{\dot{H}^s} + ||u_m||_{L^2(0,T;\dot{H}^{s+1})} + ||f_m(u)u_m||_{L^4(0,T;L^4} \le C,$$

Proof

Multiply both sides of (4.2.15) by $g_{jm}(t)$ and summing over $j = 1,...,m,$. By integration by parts, we obtain the following for each term

$$(u'_m(t), \omega_j) \cdot g_{jm}(t) = \sum_{j=1}^{3} \int u'_m g_{jm} w_j = \sum_{j=1}^{3} \int u'_m u_m dx = \sum_{j=1}^{3} \frac{1}{2} \int \frac{d}{dt}(u_m)^2 dx$$
$$\le \frac{1}{2}\frac{d}{dt}||u_m||^2$$

Similarly

$$\nu(\nabla u_m(t), \nabla \omega_j) \cdot g_{im} = \nu \sum_{j=1}^{3} \int (\nabla u_m \cdot \nabla g_{jm} w_j) dx \le \nu||\nabla u_m||^2$$

and

$$(|u_m|^2 u_m, w_j) \cdot g_{im} = \sum_{j=1}^{3} \int |u_m|^2 u_m^2 dx \le ||u_m||_{L^4}^4$$

After getting the bound on each term, we have

$$\frac{1}{2}\frac{d}{dt}||u_m||_{\dot{H}^s}^2 + \nu||\nabla u_m||_{\dot{H}^s}^2 + \beta||f(u_m)u_m||_{L^4}^4 dt \le 0$$

using $((u \cdot \nabla)v, v) = 0$. Integrate on time t over $(0,T)$, we obtain

$$sup_{0\le t\le T}||u_m||_{\dot{H}^s}^2 + 2\nu \int_0^T ||\nabla u_m||_{\dot{H}^s}^2 dt + 2\beta \int_0^T ||u_m||_{L^4}^4 dt \le ||u_0||_{\dot{H}^s}^2$$

□

Step 3

A solution of (4.2.13) is built by passing to limits as $m \to \infty$ Invoking Lemma 4.2.4,

u_m is obtained in the space $L^\infty(0,T;\dot{H}^s(\mathbb{R}^n))\cap L^2(0,T;\dot{H}^{s+1}(\mathbb{R}^n))\cap L^4(0,T;L^4(\mathbb{R}^3))$. By using Lemma 3.5.12, we prove next that u_m (or its subsequence) convergences strongly in $L^2\cap L^r([0,T]\times\mathbb{R}^3)$. $\tilde{u}_m$ is a function from $\mathbb{R}$ to $\dot{H}^{s+1}$ and on $[0,T]$, $\tilde{u}_m=u_m$, $\tilde{u}_m=0$ on $\mathbb{R}\backslash[0,T]$. In the same vein, $g_{im}(t)$ prolonged to $\mathbb{R}$. By definition, $\tilde{g}_{im}(t)=0$ for $t\in\mathbb{R}\backslash[0,T]$. The Fourier transform on variable t of $\tilde{u}_m$ and $\tilde{g}_{im}$ is given respectively by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$.

And $\tilde{u}_m$ which are the approximate solutions satisfy

$$\frac{d}{dt}(\tilde{u}_m,\omega_j)=\nu(\nabla\tilde{u}_m(t),\nabla\omega_j)+(\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t),\omega_j)+(f(u_m),\omega_j)\equiv(\tilde{f},\omega_j)+(|u_m|^2u_m,\omega_j)$$

$$j=1,2,...,m.$$

where

$$(\tilde{f}_m,\omega_j)=\nu(\nabla\tilde{u}_m(t),\nabla\omega_j)+(\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t),\omega_j).$$

Taking Fourier transform we have

$$2\pi i\tau(\hat{\tilde{u}}_m,\omega_j)=(\hat{\tilde{f}}_m(\tau),\omega_j)+\nu(f(u_m),\omega_j))+(u_{0m},\omega_j)-(u_m(T),\omega_j)exp(-2\pi iT\tau). \tag{4.2.16}$$

where $\hat{\tilde{f}}_m$ is Fourier transforms of $\tilde{f}_m$.

Multiplying (4.2.16) by $\hat{\tilde{g}}_{jm}(\tau)$ for $j=1,...,m$

$$2\pi i\tau||(\hat{\tilde{u}}_m(\tau)||_2^2=(\hat{\tilde{f}}_m(\tau),\hat{\tilde{u}}_m)+\nu(f(\hat{\tilde{u}}_m),\hat{\tilde{u}}_m))+(u_{0m},\hat{\tilde{u}}_m)-(u_m(T),\hat{\tilde{u}}_m)exp(-2\pi iT\tau). \tag{4.2.17}$$

For $v\in L^2((0,T);\dot{H}^1)\cap L^{r+1}(0,T;L^{r+1})$, we have

$$(f_m(t),v)=(\nabla u_m,\nabla v)+(u_m\cdot\nabla u_m,v)\le C(||\nabla u_m||_2^2+||\nabla u_m||_2||v||_{\dot{H}^1}$$

For $T>0$

$$\int_0^t||f_m(t)||_{\dot{H}^{-s}}dt\le\int_0^T C(||\nabla u_m||_2^2+||\nabla u_m(t)||_2)dt\le C \tag{4.2.18}$$

and hence

$$sup_{\tau\in\mathbb{R}}||\hat{\tilde{f}}_m(\tau)||_{\dot{H}^{-s}}\le\int_0^T||f_m(t)||_{\dot{H}^{-s}}\le C \tag{4.2.19}$$

Moreover, for the fact that $u\in L^4$ we have that

$$\int_0^T||u||_{L^4}^4dt\le C$$

which implies that

$$sup_{\tau\in\mathbb{R}}||\widehat{|u|^2u(\tau)}||\le C \tag{4.2.20}$$

From Lemma 4.2.4, we have

$$||u_m(0)||\le C,\ ||u_m(T)||\le C \tag{4.2.21}$$

We deduce from (4.2.18) - (4.2.21) that

$$|\tau|||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s} \leq C(||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{s+1}} + ||\hat{\tilde{u}}_m(\tau)||_{L^p}))$$

For a fixed α, $0 < \alpha < \frac{1}{4}$, it is noted that

$$|\tau|^{2\alpha} \leq C\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}},\ \forall \tau \in \mathbb{R}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{H^s} &\leq C\int_{-\infty}^{\infty}\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s}d\tau \\ &\leq C\int_{-\infty}^{\infty}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s}d\tau \\ &+ C\int_{-\infty}^{\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{s+1}}}{1+|\tau|^{1-2\alpha}}d\tau + C\int_{-\infty}^{\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{r+1}}{1+|\tau|^{1-2\alpha}}d\tau \quad (4.2.22)\end{aligned}$$

By Parseval equality as well as Lemma 4.2.4, the first integral on rhs of (4.2.22) is uniformly bounded on m.

By the Parseval equality, the Schwarz inequality and Lemma 4.2.4, we have

$$\int_{-\infty}^{+\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{s+1}}}{1+|\tau|^{1-2\alpha}}d\tau \leq \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}}\left(\int_0^T ||u_m(t)||^2_{\dot{H}^{s+1}}dt\right)^{\frac{1}{2}} \leq C \quad (4.2.23)$$

Similarly, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{||\hat{\tilde{u}}_m(\tau)||_{L^4}}{1+|\tau|^{1-2\alpha}}d\tau &\leq \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^{\frac{4}{3}}}\right)^{\frac{3}{4}}\left(\int_{-\infty}^{+\infty}||\hat{\tilde{u}}_m(\tau)||^4_{L^4}(\tau)d\tau\right)^{\frac{1}{4}} \\ &\leq C\int_{-\infty}^{+\infty}(||\hat{\tilde{u}}_m(\tau)||^4_{L^4}(\tau)d\tau)^{\frac{1}{4}} \quad (4.2.24)\end{aligned}$$

From (4.2.22), It follows that

$$\int_{-\infty}^{+\infty}|\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^s} \leq C \quad (4.2.25)$$

Hence a subsequence of u_m exists given by u_n and $u_n \rightharpoonup u$ weakly in $L^2(0,T;\dot{H}^s)$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(0,T;\dot{H}^{s+1})$. $\int_0^T\int_{\mathbb{R}^3}|u|^2udxdt \leq C$, then $u_n \longrightarrow u$ strongly in $L^p(0,T;L^p)$ for $p<\infty$. The convergences really show that $u(x,t)$ is indeed a weak solution of the BFE. $\Box$.

4.3 Weak solutions of Brinkman-Forchheimer equations with biharmonic damping term

Continuous functions need not be differentiable. In this section, existence of weak solution is obtained when the damping term $f(u)$ is a differentiable biharmonic function, $f(u) = \beta\nabla^4 u$.

$$(BFE_h)\begin{cases}\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + \beta\nabla^4 u &= 0 \qquad (t,x)\in[0,T]\times\mathbb{R}^3\\ \nabla\cdot u &= 0 \qquad \text{in } \mathbb{R}^3\\ u\mid_{t=0} &= u_0 \quad \text{in } \mathbb{R}^3\\ |u|\longrightarrow 0, \quad \text{as } |x|\longrightarrow\infty\end{cases} \tag{4.3.1}$$

The biharmonic term in this case is used to describe slow flows of viscous incompressible fluids.

The following operators can be written as

$$\mathrm{div}u = \sum_{j=1}^{3}\partial_j u^j,$$

$$u\cdot\nabla = \sum_{j=1}^{3} u^j\partial_j,$$

$$\Delta = \sum_{j=1}^{3}\partial_j^2,$$

$$\nabla^4 = \sum_{j=1}^{3}\partial_j^4$$

We still consider the whole space $\mathbb{R}^3$.

$$u\cdot\nabla u = \mathrm{div}(u\otimes u)$$

and

$$\mathrm{div}(u\otimes u)^j = \sum_{k=1}^{3}\partial_k(u^j u^k)$$

If the scalar product of (4.3.1) is taken in L^2 with the solution vector field u, we obtain

$$\frac{1}{2}||u||^2_{L^2} + (u\cdot\nabla u, u) - \nu(\Delta u, u)_{L^2} + (\nabla p, u)_{L^2} + \beta(\nabla^4 u, u)_{L^2} = 0$$

using integration by parts, we have

$$\begin{aligned}(u\nabla u \mid u)_{L^2} = \sum_{1\le j\le d}\int_{\mathbb{R}^3} u^j(\partial_j u^k)u^k dx = \frac{1}{2}\sum_{1\le j\le d}\int_{\mathbb{R}^3} u^j\partial_j(|u|^2)dx\\ = -\frac{1}{2}\sum_{1\le j\le d}\int_{\mathbb{R}^3}(\mathrm{div}|u|^2 dx = 0\end{aligned}$$

and

$$-\nu(\Delta u \mid u)_{L^2} = \nu||\nabla u||^2_{L^2}$$

,

$$-(\nabla p \mid u)_{L^2} = -\sum\int_{\mathbb{R}^3} u^j\partial_j p dx = \int_{\mathbb{R}^3} p\mathrm{div}u dx = 0$$

,

$$\beta(\nabla^4 u \mid u)_{L^2} = \beta||\Delta u||^2$$

and we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2} + \nu||\nabla u(t)||^2_{L^2} + \beta||\Delta u||^2_{L^2} = 0$$

and by integration, we obtain

$$||u(t)||^2_{L^2} + 2\nu\int_0^t ||\nabla u(t)||^2_{L^2} + 2\beta\int_0^t ||\Delta u||^2_{L^2} \leq ||u_0||^2_{L^2} \tag{4.3.2}$$

Definition 4.3.1: The function $u(x,t)$ is a weak solution of (4.3.1), given $T > 0$, the following are satisfied:

1 $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3))$

2 for any $\phi \in C_0^\infty([0,T] \times \mathbb{R}^3))$ with $\Phi(\cdot,T) = 0$, we have

$$-\int_0^T (u,\Phi_t)dt + \nu\int_0^T\int_{\mathbb{R}^3} \nabla u \cdot \nabla\Phi dxdt - \int_0^T\int_{\mathbb{R}^3}(u\cdot\nabla)u\Phi dxdt + \beta\int_0^T\int_{\mathbb{R}^3}\nabla^4 u\Phi dxdt = (u_0,\Phi_0)$$

3 div $u(x,t) = 0$ for a.e $(x,t) \in (\mathbb{R}^3 \times [0,T)$

Theorem 4.3.2:

Supposed the following conditions are satisfied:

(i) u_0 is an arbitrary function in $L^2(\mathbb{R}^3)$

(ii) $p : [0,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and
$\nabla u = (\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_i}, i = 1,2,3)$ exists

Then for $T > 0$, a weak solution $u : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ of (4.3.1) exists such that

$$u \in L^\infty\left([0,T];L^2(\mathbb{R}^3)\right) \cap L^2\left([0,T];H^1(\mathbb{R}^3)\right) \cap L^2\left([0,T],H^2(\mathbb{R}^3)\right)$$

and

$$sup_{0\leq t\leq T}||u||^2_{L^2} + 2\nu\int_0^T ||\nabla u(t)||^2_{L^2}dt + 2\beta\int_0^T ||\Delta u(t)||^2_{L^2}dt \leq ||u_0||^2_{L^2}$$

Proof:

This proof is established in sequence as follows:

Step 1

We build a weak solution in the sense of definition 4.3.1 by first constructing solutions of certain finite-dimensional approximation of (4.3.1) and the passing to limits. Since $\dot{H}^1$ is separable and C_0^∞ is dense in $\dot{H}^1$, there exists a sequence

$\omega_1, \omega_2, \omega_3, ..., \omega_m$ of members of C_0^∞, in $\dot{H}^1$. Fix a positive integer m. We look for a function $u_m : [0,T] \to \dot{H}^s(\mathbb{R}^3)$ of the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i(x) \tag{4.3.3}$$

which is an approximate solution which satisfies the equation. By multiplying the equation by a test function $w_j \in C_0^\infty$ and integrate, we obtain the following

$$(u'_m(t), \omega_j) + \nu(\nabla u_m(t), \nabla\omega_j) + (u_m(t)\cdot\nabla u_m(t), \omega_j) + (|u_m|^2 u_m, \omega_j) = 0 \tag{4.3.4}$$

$$t \in [0,T], j = 1,2,...,m. \text{ and } u_{0m} \longrightarrow u_0 \in \dot{H}^s,\ as\ m \longrightarrow \infty$$

Thus a function u_m of the form (4.3.3) is sought that satisfies the projection (4.3.4) of the problem unto the finite dimensional subspace spanned by $\{w_j\}_{j=1}^m$

Step 2

To show that a subsequence of the solutions u_m of the approximate problems converges to a weak solution of (4.3.1), uniform estimates are needed on the approximate solutions and this follows from the following Lemma.

Lemma 4.3.3:

Let $u_0 \in L^2$. Then given any $T > 0$, we have

$$sup_{0\le t\le T}||u_m||_{L^2} + ||u_m||_{L^2(0,T;\dot{H}^1)} + ||\nabla u_m||^2_{L^2(0,T;H^2)} \le C,$$

Proof

Multiply both sides of (4.3.4) by $g_{jm}(t)$ and summing over $j = 1, ..., m,$. By integration by parts, we obtain the following for each term

$$(u'_m(t), \omega_j)\cdot g_{jm}(t) = \sum_{j=1}^{3}\int u'_m g_{jm} w_j = \sum_{j=1}^{3}\int u'_m u_m dx = \sum_{j=1}^{3}\frac{1}{2}\int \frac{d}{dt}(u_m)^2 dx$$

$$\le \frac{1}{2}\frac{d}{dt}||u_m||^2$$

Similarly

$$\nu(\nabla u_m(t), \nabla\omega_j) \cdot g_{im} = \nu\sum_{j=1}^{3}\int(\nabla u_m \cdot \nabla g_{jm} w_j)dx \le \nu||\nabla u_m||^2$$

and

$$\beta(\nabla^4 u_m, w_j)\cdot g_{im} = \beta\sum_{j=1}^{3}\int_0^t(\nabla^2 u\nabla^2 u)dx \le \beta||\Delta u||^2$$

After getting the bound on each term, we have

$$\frac{1}{2}\frac{d}{dt}||u_m||^2_{L^2} + \nu||\nabla u_m||^2_{L^2} + \beta||\Delta u_m||^4_{L^4}dt \le 0$$

using $((u \cdot \nabla)v, v) = 0$. Integrate on time t over $(0, T)$, we obtain

$$sup_{0\leq t\leq T}||u_m||^2_{L^2} + 2\nu \int_0^T ||\nabla u_m||^2_{L^2} dt + 2\beta \int_0^T ||\Delta u_m||^2_{L^2} dt \leq 0$$

$\square$

Step 3

A weak solution of (4.3.1) is built by passing to limits as $m \to \infty$ Involking Lemma 4.3.3, u_m is obtained in $L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))$. By using Lemma 3.5.12, we prove that u_m convergences strongly in L^2. $\tilde{u}_m$ is denoted as a function from $\mathbb{R}$ into H^1 and on $[0, T]$, $\tilde{u}_m = u_m$ and $\tilde{u}_m = 0$ on $\mathbb{R}\backslash[0, T]$. In the same vein, $g_{im}(t)$ is extended to $\mathbb{R}$ by giving the definition $\tilde{g}_{im}(t) = 0$ for $t \in \mathbb{R}\backslash[0, T]$. The Fourier transform on variable t of $\tilde{u}_m$ and $\tilde{g}_{im}$ is given by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$ respectively.

The solutions $\tilde{u}_m$ satisfy

$$\begin{aligned}\frac{d}{dt}(\tilde{u}_m, \omega_j) &= \nu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t) \cdot \nabla\tilde{u}_m(t), \omega_j) + (\beta\nabla^4\tilde{u}_m, \omega_j)\\ &\equiv (\tilde{f}, \omega_j) + (\beta\nabla^4\tilde{u}_m, \omega_j)\\ & \qquad j = 1, 2, ..., m. \quad (4.3.5)\end{aligned}$$

where

$$(\tilde{f}_m, \omega_j) = \nu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t) \cdot \nabla\tilde{u}_m(t), \omega_j).$$

If Fourier transform is taken about the time variable (4.3.5) gives

$$\begin{aligned}2\pi i(\tau(\hat{\tilde{u}}_m, \omega_j) &= (\hat{\tilde{f}}_m(\tau), \omega_j) + \beta(\hat{\nabla^4}\tilde{u}_m, \omega_j) + (u_{0m}, \omega_j)\\ &\quad - (u_m(T), \omega_j)exp(-2\pi iT\tau). \quad (4.3.6)\end{aligned}$$

where $\hat{\tilde{f}}_m$ is Fourier transforms of $\tilde{f}_m$.

Using $\hat{\tilde{g}}_{jm}(\tau)$ to multiply (4.3.6) and add for $j = 1, ..., m$ to get:

$$\begin{aligned}2\pi i\tau||(\hat{\tilde{u}}_m(\tau)||^2_2 &= (\hat{\tilde{f}}_m(\tau), \hat{\tilde{u}}_m) + \beta(\widehat{\nabla^4\tilde{u}_m}), \hat{\tilde{u}}_m)) + (u_{0m}, \hat{\tilde{u}}_m)\\ &\quad - (u_m(T), \hat{\tilde{u}}_m)exp(-2\pi iT\tau). \quad (4.3.7)\end{aligned}$$

For any $v \in L^2$ we have

$$(f_m(t), v) = (\nabla u_m, \nabla v) + (u_m \cdot \nabla u_m, v) \leq C(||\nabla u_m||^2_2 + ||\nabla u_m||_2)||v||_{H^1}$$

Given any $T > 0$, it follows that

$$\int_0^t ||f_m(t)||_{H^{-1}} dt \leq \int_0^T C(||\nabla u_m||^2_2 + ||\nabla u_m(t)||_2) dt \leq C \qquad (4.3.8)$$

and hence

$$sup_{\tau\in\mathbb{R}}||\hat{\hat{f}}_m(\tau)||_{H^{-1}}dt \le \int_0^T ||f_m(t)||_{H^{-1}}dt \le C \qquad (4.3.9)$$

We have from Lemma 4.2.6 that

$$\int_0^T ||\nabla^4 u_m||_2 dt \le \int_0^T ||\Delta u_m||_2^2 dt \le C$$

which implies that

$$sup_{\tau\in\mathbb{R}}||\widehat{\nabla^4 u_m}||_2 \le C \qquad (4.3.10)$$

From Lemma 4.3.3, we have

$$||u_m(0)|| \le C,\ ||u_m(T)|| \le C \qquad (4.3.11)$$

We deduce from (4.3.7) - (4.3.11) that

$$|\tau|||\hat{\tilde{u}}_m(\tau)||_2^2 \le C(||\hat{\tilde{u}}_m(\tau)||_{H^1} + ||\hat{\tilde{u}}_m(\tau)||_{H^2})$$

For $0 < \alpha < \frac{1}{4}$, it is noted that

$$|\tau|^{2\alpha} \le C\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}},\ \forall\tau\in\mathbb{R}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau &\le C\int_{-\infty}^{\infty}\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau\\ &\le \int_{-\infty}^{\infty}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau + C\int_{-\infty}^{\infty}\frac{||\hat{\tilde{u}}_m(\tau)||_{H^I}}{1+|\tau|^{1-2\alpha}}\\ &\quad + C\int_{-\infty}^{\infty}\frac{||\hat{\tilde{u}}_m(\tau)|_{H^2}}{1+|\tau|^{1-2\alpha}}d\tau \qquad (4.3.12)\end{aligned}$$

By Lemma 4.3.3 and Perseval equality, the first integral on the rhs of (4.3.12) is uniformly bounded on m.

By the Parseval equality, the Schwarz inequality and Lemma 4.3.4, we have

$$\int_{-\infty}^{+\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{H^1}}{1+|\tau|^{1-2\alpha}}d\tau \le \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}}\left(\int_0^T ||u_m(t)||_{H^1}^2 dt\right)^{\frac{1}{2}} \le C \qquad (4.3.13)$$

for $0 < \alpha < \frac{1}{4}$

Also, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\tilde{\ }_m(\hat{\tau})||_{H^2}}{1+|\tau|^{1-2\alpha}}d\tau &\le \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}}\left(\int_0^T ||\hat{\tilde{u}}_m(\tau)||_{H^2}^2(\tau)d\tau\right)^{\frac{1}{2}}\\ &\le C\int_{-\infty}^{+\infty}(||\hat{\tilde{u}}_m(\tau)||_{H^2}^2 d\tau)^{\frac{1}{2}} \le C(\int_0^T ||u_m||_{H^2}^2(t)dt)^{\frac{1}{2}} \qquad (4.3.14)\end{aligned}$$

From (4.3.12)

$$\int_{-\infty}^{+\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}||_m(\tau)||_2^2 \le C \qquad (4.3.15)$$

Hence there exists a subsequence of u_m given by $u_n \ni u_n \longrightarrow u$ strongly in

$L^2(0,T;L^2)$ and $\nabla u_n \rightharpoonup \nabla u$ converges weakly in $L^2(0,T;H^1)$. $\int_0^T \int_{\mathbb{R}^3} \Delta u_n dxdt \leq C$, we obtain $\Delta u_n \rightharpoonup u$ weakly in $L^2(0,T;H^2)$. These convergences show that $u(x,t)$ is a weak solution of BFE. $\square$.

4.4 Existence of weak solutions for continuous damping term in the critical space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

The BFE is considered with the damping term $f(u) = \beta|u|^{r-1}u$. An absolute value function is continuous but not differentiable at the origin.

From the proposition 3.5.1, it is observed that if the damping term $f(u) = \beta|u|^{r-1}u$, BFE has the same scaling as NSE at the critical value of the exponent $r = 3$. If u and p solve the BFE, then the rescaled functions u_λ and p_λ also solve the equation. $\dot{H}^{\frac{1}{2}}$ and L^3 are critical spaces for BFE for $r = 3$ and their norms are conserved under the transformation $u_0 \longrightarrow u_{0,\lambda}$. In the following, we consider the existence of solutions of BFE in the critical space $\dot{H}^{\frac{1}{2}}$ for the value of the exponent $r = 3$

Theorem 4.4.1:

Suppose the following conditions are satisfied for BFE (4.1.1):

(i) u_0 is an arbitrary function in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $r = 3$

(ii) $p : [0,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and
$\nabla u = (\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_i}, i = 1,2,3)$ exists

Then for $T > 0$, a weak solution $u : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ of (4.1.1) in the sense of (1.1.11) exists such that

$$u \in L^\infty\left([0,T]; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T]; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)(\mathbb{R}^3)\right) \cap L^4\left([0,T], L^4(\mathbb{R}^3)\right)$$

and

$$sup_{0\leq t\leq T}||u||^2_{\dot{H}^{\frac{1}{2}}} \ + \ 2\nu\int_0^T ||\nabla u(t)||^2_{\dot{H}^{\frac{3}{2}}}dt \ + \ 2\beta\int_0^T ||u(t)||^4_{L^4}dt \quad \leq \quad ||u_0||^2_{\dot{H}^{\frac{1}{2}}}$$

Proof

This proof is established in sequence as follows:

Step 1

We build a weak solution in the sense of (1.1.11) by first constructing solutions u_m of finite-dimensional approximation of (4.1.1) and pass to limits. Since $\dot{H}^{\frac{1}{2}}$ is separable and C_0^∞ is dense in $\dot{H}^{\frac{1}{2}}$, there exists a sequence $\omega_1, \omega_2, \omega_3, ..., \omega_m$ of members of C_0^∞, in $\dot{H}^{\frac{1}{2}}$. Fix a positive integer m. We look for a function u_m :

$[0,T] \to \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ of the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i(x) \tag{4.4.1}$$

which is an approximate solution which satisfies the equation. By multiplying the equation by a test function $w_j \in C_0^\infty$ and integrate, we obtain the following

$$(u'_m(t), \omega_j) + \nu(\nabla u_m(t), \nabla\omega_j) + (u_m(t)\cdot\nabla u_m(t), \omega_j) + (|u_m|^2 u_m, \omega_j) = 0 \tag{4.4.2}$$

$$t \in [0,T], j = 1,2,...,m. \text{ and } u_{0m} \longrightarrow u_0 \in \dot{H}^{\frac{1}{2}},\ as\ m \longrightarrow \infty$$

Thus a function u_m of the form (4.4.1) is sought and satisfies (4.4.2) spanned by $\{w_j\}_{j=1}^m$ unto the finite dimensional subspace

Step 2

To show that a subsequence of the solutions u_m converges to a weak solution of (4.1.1), uniform estimates are needed on the approximate solutions and this follows from the following Lemma.

Lemma 4.4.2:

Let $u_0 \in \dot{H}^{\frac{1}{2}}$. Then for any $T > 0$, we have

$$sup_{0\le t\le T}||u_m||_{\dot{H}^{\frac{1}{2}}} + ||u_m||_{L^2(0,T;\dot{H}^{\frac{3}{2}}} + ||u_m||^4_{L^4(0,T;^4} \le C,$$

Proof

Multiply both sides of (4.4.2) by $g_{jm}(t)$ and summing over $j = 1,...,m,$. By integration byL parts, we obtain the following for each term

$$(u'_m(t), \omega_j)\cdot g_{jm}(t) = \sum_{j=1}^{3}\int u'_m g_{jm} w_j = \sum_{j=1}^{3}\int u'_m u_m dx = \sum_{j=1}^{3}\frac{1}{2}\int \frac{d}{dt}(u_m)^2 dx$$

$$\le \frac{1}{2}\frac{d}{dt}||u_m||^2$$

Similarly

$$\nu(\nabla u_m(t), \nabla\omega_j) \cdot g_{im} = \nu\sum_{j=1}^{3}\int(\nabla u_m \cdot \nabla g_{jm} w_j)dx \le \nu||\nabla u_m||^2$$

and

$$(|u_m|^2 u_m, w_j) \cdot g_{im} = \sum_{j=1}^{3}\int |u_m|^2 u_m^2 dx \le ||u_m||^4_{L^4}$$

After getting the bound on each term, we have

$$\frac{1}{2}\frac{d}{dt}||u_m||^2_{\dot{H}^s} + \nu||\nabla u_m||^2_{\dot{H}^s} + \beta||f(u_m)u_m||^4_{L^4}dt \le 0$$

using $((u\cdot\nabla)v, v) = 0$. Integrate on time t over $(0,T)$, we obtain

$$sup_{0\le t\le T}||u_m||^2_{\dot{H}^s} + 2\nu\int_0^T ||\nabla u_m||^2_{\dot{H}^s}dt + 2\beta\int_0^T ||u_m||^4_{L^4}dt \le ||u_0||^2_{\dot{H}^s}$$

$\square$

Step 3

A solution of (4.1.1) is built by passing to limits as $m \longrightarrow \infty$. Invoking Lemma 4.3.2, the solutions u_m is obtained in

$$L^\infty(0,T;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^4(0,T;L^4(\mathbb{R}^3))$$

and strong convergence of u_m is proved in $L^2 \cap L^4([0,T] \times \mathbb{R}^3)$. To this end, we denote $\tilde{u}_m$ taking value from $\mathbb{R}$ into $\dot{H}^{\frac{3}{2}}$ and has the same value as u_m on $[0,T]$ and zero on it's complement . I the same vein, $g_{im}(t)$ is extended to $\mathbb{R}$ and $\tilde{g}_{im}(t) = 0$ is defined for $t \in \mathbb{R}\backslash[0,T]$. The Fourier transform of $\tilde{u}_m$ and $\tilde{g}_{im}$ is given by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$ respectively on time variable.

Approximate solutions $\tilde{u}_m$ satisfy

$$\frac{d}{dt}(\tilde{u}_m, \omega_j) = \nu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t), \omega_j) + (\beta|\tilde{u}_m|^2\tilde{u}_m(t), \omega_j) \equiv$$
$$(\tilde{f}, \omega_j) + (\beta|\tilde{u}_m|^2\tilde{u}_m(t), \omega_j)$$
$$j = 1, 2, ..., m.$$

where

$$(\tilde{f}_m, \omega_j) = \nu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t), \omega_j).$$

If the Fourier transform is taken about the time variable, we obtain

$$2\pi i(\tau(\hat{\tilde{u}}_m, \omega_j) = (\hat{\tilde{f}}_m(\tau), \omega_j) + \nu(|\tilde{u}_m|^2\widehat{\tilde{u}_m(\tau)}, \omega_j)) + (u_{0m}, \omega_j) - (u_m(T), \omega_j)exp(-2\pi iT\tau). \tag{4.4.3}$$

where $\hat{\tilde{f}}_m$ denote the Fourier transforms of $\tilde{f}_m$.

Multiplying (4.4.3) by $\hat{\tilde{g}}_{jm}(\tau)$, we obtain:

$$2\pi i\tau||(\hat{\tilde{u}}_m(\tau)||_2^2 = (\hat{\tilde{f}}_m(\tau), \hat{\tilde{u}}_m) + \nu(|\tilde{u}_m|^2\widehat{\tilde{u}_m(\tau)}, \hat{\tilde{u}}_m)) + (u_{0m}, \hat{\tilde{u}}_m) - (u_m(T), \hat{\tilde{u}}_m)exp(-2\pi iT\tau). \tag{4.4.4}$$

For $v \in L^2((0,T);\dot{H}^{\frac{3}{2}}) \cap L^4(0,T;L^4)$, we have

$$(f_m(t), v) = (\nabla u_m, \nabla v) + (u_m\cdot\nabla u_m, v) \le C(||\nabla u_m||^2_{\dot{H}^{\frac{1}{2}}} + ||\nabla u_m||_2||v||_{\dot{H}^{\frac{1}{2}}}$$

Given any $T > 0$, It follows that

$$\int_0^t ||f_m(t)||_{\dot{H}^{-\frac{1}{2}}}dt \le \int_0^T C(||\nabla u_m||^2_{\dot{H}^{\frac{1}{2}}} + ||\nabla u_m(t)||_{H^{\frac{1}{2}}})dt \le C \tag{4.4.5}$$

and hence

$$sup_{\tau\in\mathbb{R}}||\hat{\tilde{f}}_m(\tau)||_{\dot{H}^{-\frac{1}{2}}} \le \int_0^T ||f_m(t)||_{\dot{H}^{-\frac{1}{2}}} \tag{4.4.6}$$

Moreover, we have that

$$\int_0^T |||u_m|^2u_m||dt \le \int_0^T ||u_m||dt \le C$$

which implies that

$$sup_{\tau\in\mathbb{R}}||\widehat{|u_m|^{r-1}u}|| \leq C \tag{4.4.7}$$

From Lemma 4.4.2, we have

$$||u_m(0)|| \leq C, \ ||u_m(T)|| \leq C \tag{4.4.8}$$

We deduce from (4.4.3) - (4.4.7) that

$$|\tau|||\hat{\tilde{u}}_m(\tau)||^2_{\dot{H}^{\frac{1}{2}}} \leq C(||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{\frac{3}{2}}} + ||\hat{\tilde{u}}_m(\tau)||_{L^4})$$

For $0 < \alpha < \frac{1}{4}$, it is observed that

$$|\tau|^{2\alpha} \leq C\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}, \ \forall\tau\in\mathbb{R}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty}|\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{\frac{1}{2}}} &\leq C\int_{-\infty}^{\infty}\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{\frac{1}{2}}}d\tau \\ &\leq \int_{-\infty}^{\infty}||\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{\frac{1}{2}}}d\tau + C\int_{-\infty}^{\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{\frac{3}{2}}}}{1+|\tau|^{1-2\alpha}} \\ &\quad + C\int_{-\infty}^{\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{L^4}}{1+|\tau|^{1-2\alpha}}d\tau \end{aligned} \tag{4.4.9}$$

By Parseval equality and Lemma 4.4.2, the first integral on the rhs of the (4.4.9) is uniformly bounded on m.

By the Schwarz inequality, the Parseval equality and Lemma 4.4.2, we obtain

$$\int_{-\infty}^{+\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{\dot{H}^{\frac{3}{2}}}}{1+|\tau|^{1-2\alpha}}d\tau \leq \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}}\left(\int_0^T||u_m(t)||^2_{\dot{H}^{\frac{3}{2}}}dt\right)^{\frac{1}{2}} \tag{4.4.10}$$

for $0 < \alpha < \frac{1}{4}$

Similarly, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\hat{\tilde{u}}_m(\tau)||_{L^4}}{1+|\tau|^{1-2\alpha}}d\tau &\leq \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\alpha})^{\frac{4}{3}}}\right)^{\frac{3}{4}}\left(\int_{-\infty}^{+\infty}||\hat{\tilde{u}}_m(\tau)||^4_{L^4}(\tau)d\tau\right)^{\frac{1}{4}} \\ &\leq C\int_{-\infty}^{+\infty}(||\hat{\tilde{u}}_m(\tau)||^{\frac{4}{3}}_{L^4}(\tau)d\tau)^{\frac{3}{4}} \\ &\quad C(\int_0^T||u_m||^4_{L^4}(t)dt)^{\frac{1}{4}}\end{aligned} \tag{4.4.11}$$

It follows from (4.4.9) that

$$\int_{-\infty}^{+\infty}|\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)|^2|_{\dot{H}^{\frac{1}{2}}} \leq C \tag{4.4.12}$$

Hence there exists a subsequence of u_m given by u_n such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;\dot{H}^{\frac{1}{2}})$ and $\nabla u_n \rightharpoonup \nabla u$ converges weakly in $L^2(0,T;\dot{H}^{\frac{3}{2}})$. Note that $\int_0^T\int_{\mathbb{R}^3}|u|^2udxdt \leq C$, we obtain $u_n \longrightarrow u$ strongly in $L^4(0,T;L^4)$. These convergences guarantee that $u(x,t)$ is a weak solution of the BFE. □.

4.5 Existence of Weak Solutions of BFE with Biharmonic damping term in the Critical Space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

The previous result concerning the continuously differentiable biharmonic term is proved in square integrable space L^2. The following results show that the solution can also exist in the critial homgeneous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

Theorem 4.5.1:

Supposed the following conditions are satisfied:

(i) u_0 is an arbitrary function in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

(ii) $p : [0, T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and
$\nabla u = (\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x_i}, i = 1, 2, 3)$ exists

Then for $T > 0$, a weak solution $u : [0, T] \times \mathbb{R}^3$ of (4.2.1) in the sense of (1.1.11) exists such that

$$u \in L^\infty\left([0, T]; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0, T]; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0, T], \dot{H}^{\frac{5}{2}}(\mathbb{R}^3)\right)$$

and

$$sup_{0\leq t\leq T}||u||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\nu \int_0^T ||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} dt + 2\beta \int_0^T ||\Delta u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} dt \leq ||u_0||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$$

Proof:

This proof is established in sequence as follows:

Step 1

A weak solution is built in the sense of definition (4.2.5) by first constructing solutions u_m of finite-dimensional approximation of (4.2.1) and the passing to limits. $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ being separable and since C_0^∞ is dense in it, there is a sequence $\omega_1, \omega_2, \omega_3, ..., \omega_m$ of members of C_0^∞, in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Fix a positive integer m. We look for a function $u_m : [0, T] \to \dot{H}^s(\mathbb{R}^3)$ of the form

$$u_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i(x) \tag{4.5.1}$$

which is an approximate solution which satisfies the equation. By multiplying the equation by a test function $w_j \in C_0^\infty$ and integrate, we obtain the following

$$(u'_m(t), \omega_j) + \nu(\nabla u_m(t), \nabla\omega_j) + (u_m(t) \cdot \nabla u_m(t), \omega_j) + (|u_m|^2 u_m, \omega_j) = 0 \tag{4.5.2}$$

$$t \in [0, T], j = 1, 2, ..., m. \text{ and } u_{0m} \longrightarrow u_0 \in \dot{H}^s, \ as\ m \longrightarrow \infty$$

Thus a function u_m of the form (4.5.1) is sought and satisfies (4.5.2) spanned by $\{w_j\}_{j=1}^m$ projected unto the finite dimensional subspace

Step 2

To show that a subsequence of the solutions u_m converges to a weak solution of (4.2.1), uniform estimates are needed on the approximate solutions and this follows from the following Lemma.

Lemma 4.5.2:

Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then given any $T > 0$, we have

$$sup_{0\leq t\leq T}||u_m||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + ||u_m||_{L^2(0,T;\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} + ||\nabla u_m||^2_{L^2(0,T;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \leq C,$$

Proof

Multiply both sides of (4.5.2) by $g_{jm}(t)$ and summing over $j = 1, ..., m,$. By integration by parts, we obtain the following for each term

$$(u'_m(t), \omega_j) \cdot g_{jm}(t) = \sum_{j=1}^{3} \int u'_m g_{jm} w_j = \sum_{j=1}^{3} \int u'_m u_m dx = \sum_{j=1}^{3} \frac{1}{2} \int \frac{d}{dt}(u_m)^2 dx$$
$$\leq \frac{1}{2}\frac{d}{dt}||u_m||^2$$

Similarly

$$\nu(\nabla u_m(t), \nabla \omega_j) \cdot g_{im} = \nu \sum_{j=1}^{3} \int (\nabla u_m \cdot \nabla g_{jm} w_j) dx \leq \nu||\nabla u_m||^2$$

and

$$\beta(\nabla^4 u_m, w_j) \cdot g_{im} = \beta \sum_{j=1}^{3} \int_0^t (\nabla^2 u \nabla^2 u) dx \leq \beta||\Delta u||^2$$

After getting the bound on each term, we have

$$\frac{1}{2}\frac{d}{dt}||u_m||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \nu||\nabla u_m||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \beta||\Delta u_m||^4_{L^4} dt \leq 0$$

using $((u \cdot \nabla)v, v) = 0$. Integrate on time t over $(0, T)$, we obtain

$$sup_{0\leq t\leq T}||u_m||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\nu \int_0^T ||\nabla u_m||^2_{L^2} dt + 2\beta \int_0^T ||\Delta u_m||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} dt \leq C$$

□

Step 3

A solution of (4.2.1) is then built by passing to limits as $m \to \infty$ Involking Lemma 4.5.2, the solutions u_m is obtained in the space $L^\infty(0, T; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(0, T; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))$. By using Lemma 3.5.12, we prove that u_m convergences strongly in L^2. $\tilde{u}_m$ is denoted as a function from $\mathbb{R}$ into H^1 and $\tilde{u}_m = u_m$ on $[0, T]$ and $\tilde{u}_m = 0$ on it's complement. In the same vein, $g_{im}(t)$ is extended to $\mathbb{R}$ by giving the definition $\tilde{g}_{im}(t) = 0$ for $t \in \mathbb{R}\backslash[0, T]$. The Fourier transform on variable t of

$\tilde{u}_m$ and $\tilde{g}_{im}$ is given by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$ respectively.

The solutions $\tilde{u}_m$ satisfy

$$\begin{aligned}\frac{d}{dt}(\tilde{u}_m, \omega_j) = \nu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t), \omega_j) + (\beta\nabla^4\tilde{u}_m, \omega_j) \\ \equiv (\tilde{f}, \omega_j) + (\beta\nabla^4\tilde{u}_m, \omega_j) \\ j = 1, 2, ..., m. \quad (4.5.3)\end{aligned}$$

where

$$(\tilde{f}_m, \omega_j) = \nu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t), \omega_j).$$

If Fourier transform is taken about the time variable (4.5.3) gives

$$\begin{aligned}2\pi i(\tau(\hat{\tilde{u}}_m, \omega_j) = (\hat{\tilde{f}}_m(\tau), \omega_j) + \beta(\hat{\nabla^4}\tilde{u}_m, \omega_j) + (u_{0m}, \omega_j) \\ - (u_m(T), \omega_j)exp(-2\pi iT\tau). \quad (4.5.4)\end{aligned}$$

where $\hat{\tilde{f}}_m$ is Fourier transforms of $\tilde{f}_m$.

Using $\hat{\tilde{g}}_{jm}(\tau)$ to multiply (4.5.4) and add for $j = 1, ..., m$ to get:

$$\begin{aligned}2\pi i\tau||(\hat{\tilde{u}}_m(\tau)||_2^2 = (\hat{\tilde{f}}_m(\tau), \hat{\tilde{u}}_m) + \beta(\widehat{\nabla^4\tilde{u}}_m), \hat{\tilde{u}}_m)) + (u_{0m}, \hat{\tilde{u}}_m) \\ - (u_m(T), \hat{\tilde{u}}_m)exp(-2\pi iT\tau). \quad (4.5.5)\end{aligned}$$

For any $v \in L^2$ we have

$$(f_m(t), v) = (\nabla u_m, \nabla v) + (u_m\cdot\nabla u_m, v) \leq C(||\nabla u_m||_2^2 + ||\nabla u_m||_2)||v||_{H^1}$$

Given any $T > 0$, it follows that

$$\int_0^t ||f_m(t)||_{H^{-1}}dt \leq \int_0^T C(||\nabla u_m||_2^2 + ||\nabla u_m(t)||_2)dt \leq C \quad (4.5.6)$$

and hence

$$sup_{\tau\in\mathbb{R}}||\hat{\tilde{f}}_m(\tau)||_{H^{-1}}dt \leq \int_0^T ||f_m(t)||_{H^{-1}}dt \leq C \quad (4.5.7)$$

We have from Lemma 4.5.2 that

$$\int_0^T ||\nabla^4 u_m||_2 dt \leq \int_0^T ||\Delta u_m||_2^2 dt \leq C$$

which implies that

$$sup_{\tau\in\mathbb{R}}||\widehat{\nabla^4 u_m}||_2 \leq C \quad (4.5.8)$$

From Lemma 4.5.2, we have

$$||u_m(0)|| \leq C, \ ||u_m(T)|| \leq C \quad (4.5.9)$$

We deduce from (4.5.5) - (4.5.9) that

$$|\tau|||\hat{\tilde{u}}_m(\tau)||_2^2 \leq C(||\hat{\tilde{u}}_m(\tau)||_{H^1} + ||\hat{\tilde{u}}_m(\tau)||_{H^2})$$

For $0 < \alpha < \frac{1}{4}$, it is noted that

$$|\tau|^{2\alpha} \leq C\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}, \ \forall\tau \in \mathbb{R}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau &\leq C\int_{-\infty}^{\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau \\ &\leq \int_{-\infty}^{\infty} ||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau + C\int_{-\infty}^{\infty} \frac{||\hat{\tilde{u}}_m(\tau)||_{H^I}}{1+|\tau|^{1-2\alpha}} \\ &+ C\int_{-\infty}^{\infty} \frac{||\hat{\tilde{u}}_m(\tau)|_{H^2}}{1+|\tau|^{1-2\alpha}} d\tau \quad (4.5.10)\end{aligned}$$

By Lemma 4.5.2 and Perseval equality, the first integral on the rhs of (4.5.10) is uniformly bounded on m.

By the Parseval equality, the Schwarz inequality and Lemma 4.5.2, we have

$$\int_{-\infty}^{+\infty} \frac{\hat{\tilde{u}}_m(\tau)||_{H^1}}{1+|\tau|^{1-2\alpha}} d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}} \left(\int_0^T ||u_m(t)||_{H^1}^2 dt\right)^{\frac{1}{2}} \leq C \quad (4.5.11)$$

for $0 < \alpha < \frac{1}{4}$

Also, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\tilde{\ }_m(\hat{\tau})||_{H^2}}{1+|\tau|^{1-2\alpha}} d\tau &\leq \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}} \left(\int_0^T ||\hat{\tilde{u}}_m(\tau)||_{H^2}^2(\tau) d\tau\right)^{\frac{1}{2}} \\ &\leq C\int_{-\infty}^{+\infty} (||\hat{\tilde{u}}_m(\tau)||_{H^2}^2 d\tau)^{\frac{1}{2}} \leq C(\int_0^T ||u_m||_{H^2}^2(t)dt)^{\frac{1}{2}} \quad (4.5.12)\end{aligned}$$

and

$$\int_{-\infty}^{+\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}||_m(\tau)||_2^2 \leq C \quad (4.5.13)$$

Hence there exists a subsequence of u_m given by $u_n \ni u_n \longrightarrow u$ strongly in $L^2(0,T;L^2)$ and $\nabla u_n \rightharpoonup \nabla u$ converges weakly in $L^2(0,T;H^1)$. $\int_0^T \int_{\mathbb{R}^3} \Delta u_n dxdt \leq C$, we obtain $\Delta u_n \rightharpoonup u$ weakly in $L^2(0,T;H^2)$. These convergences show that $u(x,t)$ is a weak solution of BFE. □.

4.6 The Main Theorem

In the previous sections, existence results of weak solutions were obtained. Some of these results were obtained in the critical homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ where the value of the exponent $r = 3$ when the damping term $f(u) = |u|^{r-1}u$ is considered. This section deals with the stability of BFE (4.1.1) with respect to initial data in the critical homogeneous spaces. Profile decomposition method is employed to obtain our results in this section.

In accordance with the existence results obtained in the previous sections, we define the function spaces for BFE when $u_0 \in \dot{H}^{\frac{1}{2}}$.

$$\begin{cases} E_T &= C^0\left([0,T],\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right)\cap L^2\left([0,T],\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right)\cap L^4\left([0,T],L^4(\mathbb{R}^3)\right) \\ E_\infty &= C^0_b\left([\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right)\cap L^2\left(\mathbb{R}^+,\dot{H}^{\frac{3}{2}}(\mathbb{R}^+)\right)\cap L^4\left(\mathbb{R}^+,L^4(\mathbb{R}^3)\right) \end{cases} \tag{4.6.1}$$

where C^0_b is the set of bounded and continuous functions and set of initial data yielding solutions of (BFE) in E_T and E_∞ are

$$I\!D_T = \{u_0 \in \dot{H}^s(\mathbb{R}^3) | BF(u_0) \in E_T\}$$

$$I\!D_\infty = \{u_0 \in \dot{H}^s(\mathbb{R}^3) | BF(u_0) \in E_\infty\}$$

and we define, for any vector field u,

$$\begin{cases} ||u||_{E_T} = & \sup_{0\le t\le T}\left(||u(t)||^2_{\dot{H}^s(\mathbb{R}^3)} + 2\nu||u(t)||^2_{L^2([0,t],\dot{H}^{s+1}(\mathbb{R}^3))} + 2\beta||u||^2_{L^4([0,T],L^4(\mathbb{R}^3))}\right)^{\frac{1}{2}} \\ & ||u||_{E_\infty} = \left(||u||^2_{L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} + 2\nu||u||^2_{L^2(\mathbb{R}^+,\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} + 2\beta||u||^2_{L^4(\mathbb{R}^+,L^4(\mathbb{R}^3)}\right)^{\frac{1}{2}} \end{cases} \tag{4.6.2}$$

The theorem involves the sequences of solutions of the critical BFE in $\mathbb{R}^3$ together with bounded sequences of initial data in $\dot{H}^{\frac{1}{2}}$. It is shown that the sequence can be decomposed into a sum of bounded orthogonal profiles in $\dot{H}^{\frac{1}{2}}$, to a small remainder term in L^3. Also, since L^3 is an admissible space according to Definition 3.3.2, if B^A_{BF} is a ball in L^3 with center zero such that the members of $\dot{H}^{\frac{1}{2}} \cap B^A_{BF}$ generate global solutions, then an a priori estimate is obtained for those solutions.

Theorem 4.6.1:

Suppose the following conditions hold:

i (φ_n) is a collection of bounded and divergence free vector fields in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

ii $\varphi^0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is a weak limit point of (φ_n)

iii A is an admissible space and $\rho \in [0, C^A_{BF}]$ is a real number where C^A_{BF} is given by equation (3.5.9)

Then the following results hold:

i There exists a collection $(T^j)_{j\in\mathbb{N}}$ of members of $\mathbb{R}^+ \cup \{+\infty\}$ and a finite $J \subset \mathbb{N}$ such that

$$V^j \in E_{T^j}\ \forall\, j \in \mathbb{N} \text{ and } \quad T^j = +\infty\ \forall\, j \in \mathbb{N}\setminus J \tag{4.6.3}$$

ii If $||\varphi_n||_A \leq \rho$, then $T^j = +\infty$ for every $j \in \mathbb{N}$ and there exists a nondecreasing function $B : \mathbb{R}^+ \times [0, C^A_{BF}] \longrightarrow \mathbb{R}^+$ such that for $\varphi \in B^A_{BF}$, we have

$$||BF(\varphi)||_{E_\infty} \leq B(||\varphi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, ||\varphi||_A)$$

Remark 4.6.2:

Existence theorems are usually proved by establishing some a priori estimate. The theorem reveals that whichever method employed to obtain global existence B^A_{BF} , there is an a priori estimate for those solutions.

Before the proof of Theorem 4.6.1, some orthogonality results are proved concerning the BFE having initial data as profiles.

The BFE with profiles as initial data

Given initial data of the form

$$\varphi_n = \frac{1}{h_n}\varphi\left(\frac{x - x_n}{h_n}\right),$$

with $(h_n, x_n) \in (\mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^3)^{\mathbb{N}}$. The following proposition gives some orthogonal results for BFE

Proposition 4.6.3:

Given any $T \in \mathbb{R}^+ \cup \{+\infty\}$, and let φ^1 and φ^2 be vector fields members of $I\!D_T$ which are divergence free . Let orthogonal sequences (h^1_n, x^1_n) and (h^2_n, x^2_n) of $(\mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^3)^{\mathbb{N}}$ be considered in accordance with (3.3.1). Suppose for instance that $h^1_n \leq h^2_n$. Then with notation (3.3.5), we have the following orthogonality results:

$$\lim_{n\to\infty} sup_{t\in[0,(h^1_n)^2T]} \left(BF(\varphi^1_n)(t,\cdot)|BF(\varphi^2_n)(t,\cdot)\right)_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = 0 \tag{4.6.4}$$

as well as

$$\lim_{n\to\infty} sup_{t\in[0,(h^1_n)^2T]} \left(BF(\varphi^1_n)|BF(\varphi^2_n)(\right)_{L^2([0,(h^1_n)^2T],\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} = 0 \tag{4.6.5}$$

and

$$\lim_{n\to\infty} ||BF(\varphi^1_n)BF(\varphi^2_n)||_{L^4([0,(h^1_n)^2T],L^2(\mathbb{R}^3))} = 0 \tag{4.6.6}$$

Proof:

By scale-invariance property of BFE, the solution of BF with the data φ^j_n is given by

$$\forall\ j \in \{1,2\}, u^j_n(t,x) = BF(\varphi^j_n)(t,x) = \frac{1}{h^j_n}V^j\left(\frac{t}{(h^j_n)^2}, \frac{x - x^j_n}{h^j_n}\right),$$

where $V^j = BF(\varphi^j)$. Note that $V^j \in E_T$, so $u^j_n \in E_{(h^j_n)^2T}$. The result does not mean V^j solves (BFE), we need $V^j \in E_T$; so the functions V^j are assumed be smooth and compact support, then we obtain

$$\begin{aligned}&\big(BF(\varphi_n^1)(t,\cdot)|BF(\varphi_n^2)(t,\cdot)\big)_{\dot H^{\frac12}(\mathbb{R}^3)}\\&\quad=\int_{\mathbb{R}^3}(h_n^1)^{-\frac32}(h_n^2)^{-\frac32}(\Lambda^{\frac12}V^1)\left(\frac{t}{(h_n^1)^2},\frac{x-x_n^1}{h_n^1}\right)(\Lambda^{\frac12}V^2)\left(\frac{t}{(h_n^2)^2},\frac{x-x_n^2}{h_n^2}\right)dx\end{aligned}$$

and $\Lambda=\sqrt{-\Delta}$. Assume that $\lim_{n\to\infty}\frac{h_n^1}{h_n^2}=0$. Then the change of variables

$$x=x_n^1+h_n^1y,\ t=(h_n^1)^2s \tag{4.6.7}$$

yields

$$\forall\, t\in[0,(h_n^1)^2T],\quad \left|\big(BF(\varphi_n^1)(t,\cdot)|BF(\varphi_n^2)(t,\cdot)\big)_{\dot H^{\frac12}(\mathbb{R}^3)}\right|=O(\frac{h_n^1}{h_n^2})^{\frac32},$$

which gives result. The sequences (h_n^1,x_n^1) and (h_n^2,x_n^2) are orthogonal, in the sense of (3.3.1), and we have supposed that $h_n^1\le h_n^2$. So if $\lim_{n\to\infty}\frac{h_n^1}{h_n^2}\neq0$, then $h_n^1=h_n^2$. In that case, the change of variables (4.6.7) gives the estimate for $t\ge0$ as follows,

$$\big(BF(\varphi_n^1)(t,\cdot)|BF(\varphi_n^2)(t,\cdot)\big)_{\dot H^{\frac12}(\mathbb{R}^3)}=\int_{\mathbb{R}^3}(\Lambda^{\frac12}V^1)(s,y)(\Lambda^{\frac12}V^2)\left(s,y+\frac{x_n^1-x_n^2}{h_n^1}\right)dx \tag{4.6.8}$$

The result (4.6.4) follows from the orthogonality property (3.3.1) since it is assumed that V^2 has a compact support. The arguments are the same for (4.6.5). To prove (4.6.6). Assume that $\lim_{n\to\infty}\frac{h_n^1}{h_n^2}=0$. Then for any $t\in[0,(h_n^1)^2T]$, we have

$$\begin{aligned}||BF(\varphi_n^1)BF(\varphi_n^2)||^4_{L^4([0,t],L^2(\mathbb{R}^3)}&=(h_n^1h_n^2)^{-2}\\&\times\int_0^t\{\int_{\mathbb{R}^3}\left|V^1\left(\frac{t}{(h_n^1)^2},\frac{x-x_n^1}{h_n^1}\right)\right|^2\\&\qquad\times\left|V^2\left(\frac{t}{(h_n^2)^2},\frac{x-x_n^2}{h_n^2}\right)\right|^2dx\}^2dt\end{aligned}$$

Then the change of variables (4.6.7) yields

$$||u_n^1u_n^2|||_{L^4([0,t],L^2(\mathbb{R}^3))}=O(\frac{h_n^1}{h_n^2}),$$

then the result follows,

For the case when $h_n^1=h_n^2$, the change of variable implies that

$$||u_n^1u_n^2||^4_{L^4([0,(h_n^1)^2T],L^2(\mathbb{R}^3))}=\int_0^T\{\int_{\mathbb{R}^3}\left|V^1(s,y)\right|^2\times\left|V^2\left(s,y+\frac{x_n^1-x_n^2}{h_n^2}\right)\right|^2dy\}^2ds$$

and since V^2 has a compact support, we obtain result. Then the proposition is proved □

Proof of Theorem 4.6.1

i Let us first prove property (4.6.3) by writing the decomposition (3.3.2) of theorem 3.3.2 that reads

$$\forall\,\ell\in\mathbb{N},\quad\forall\,n\in\mathbb{N},\quad\varphi_n(x)=\sum_{j=0}^{\ell}\varphi_n^j(x)+\psi_n^\ell(x) \tag{4.6.9}$$

with notation (3.3.5) and (3.3.6) and with properties (3.3.1), (3.3.3) and (3.3.4). Then by the limit (3.3.3), we are enabled to define an integerℓ_0 such that

$$\lim_{n\to\infty} sup||\psi_n^{\ell_0}||_{L^3(\mathbb{R}^3)} \leq c\nu \tag{4.6.10}$$

Then in particular

$(BF(\psi_n^{\ell_0}))$ is bounded in E_∞

Since $||\psi_n^{\ell_0}||_{\dot{H}^{\frac{1}{2}}}$ is bounded according to (3.3.4). Since L^3 is an admissible space according to definition (3.5.9). This implies that any function in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ whose L^3 norm is smaller than $c\nu$ is contained in $I\!D_\infty$, so we have that

$$c\nu < C_{BF}^{L^3}$$

with the notation (3.5.9). So the result of Theorem 4.6.1(ii) can be applied which gives

$$BF(\psi_n^{\ell_0})(t,x) = \sum_{j=\ell_0+1}^{\ell} \frac{1}{h_n^j} V^j \left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right) + w_n^\ell(t,x) + \tilde{r}_n^\ell(t,x) \tag{4.6.11}$$

with the orthogonality property (3.3.1), the limit (3.5.5) for w_n^ℓ and where we have noted $V^j = BF(\varphi^j)$ for $j \geq \ell_0 + 1$. In addition V^j is a member of E_∞ and

$$\lim_{\ell\to\infty} \lim_{n\to\infty} sup\tilde{r}_n^\ell = 0 \text{ in } E_\infty \tag{4.6.12}$$

That proves the property (4.6.3), choosing $J = \{0, ..., \ell_0\}$

ii Let A be an admissible space according to Definition 3.5.9, and let $\rho \in [0, C_{BF}^A]$ be given such that

$$\forall\ n \in \mathbb{N}, \quad ||\varphi_n||_A \leq \rho \tag{4.6.13}$$

For every $j \in \mathbb{N}$, with Theorem (3.3.2) notation, the function φ^j is a weak limit point in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ of the sequence

$$\Phi_n^j = h_n^j \varphi_n(x_n^j + h_n^j)$$

By property (i) of Definition 3.5.9, we have

$$\forall\ (j,n) \in \mathbb{N}^2, \quad ||\varphi_n||_A = ||\Phi_n^j||_A \tag{4.6.14}$$

Moreover, since $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is embedded in A, we have $\varphi^j \in A$, and because the space A is Banach, (4.6.13) and (4.6.14) imply that

$$\forall\ j \in \mathbb{N}, \quad ||\varphi^j||_A \leq \rho$$

Following the definition of ρ, we have $\varphi^j \in I\!\!D_\infty$ $\forall$ integers j, and as a result, a global solution of the BFE can be associated with φ^j.

$$\forall\ j \in \mathbb{N}, \quad V^j = BF(\varphi^j)\ \in E_\infty, \tag{4.6.15}$$

where recall that

$$E_\infty = C_b^0\left(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \cap L^4\left(\mathbb{R}^+, L^4(\mathbb{R}^3)\right)$$

By the scale invariance of the BFE in proposition (3.5.1), it is known that

$$u_n^j(t,x) = \frac{1}{h_n^j} V^j\left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right)$$

is the global solution which is unique with associated data φ_n^j. For every integer $\ell \in \mathbb{N}$, we define

$$r_n^\ell = u_n - \sum_{j\leq \ell} u_n^j - \omega_n^\ell, \tag{4.6.16}$$

where $\omega_n^\ell = H(\psi_n^\ell)$ is uniformly bounded in E_∞ and $\ell \in \mathbb{N}$, with

$$\lim_{\ell\to\infty}\left(\lim_{n\to\infty}\ sup||\omega_n^\ell||_{L^\infty(\mathbb{R}^+, L^3))}\right) = 0 \tag{4.6.17}$$

and $u_n = BF(\psi_n)$. To obtain the result,it is sufficient to prove that

$$\lim_{\ell\to\infty}(\lim_{n\to\infty} \sup r_n^\ell) = 0 \text{ in } E_\infty. \tag{4.6.18}$$

The function r_n^ℓ satisfies the following system:

$$\begin{cases} \partial_t r_n^\ell + P(r_n^\ell \cdot \nabla r_n^\ell) - \nu\Delta r_n^\ell + Q(r_n^\ell, f_n^\ell) & = & g_n^\ell \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ r_{n|t=0}^\ell & = & 0 \end{cases} \tag{4.6.19}$$

where

$$f_n^\ell = \sum_{j\leq\ell} u_n^j + \omega_n^\ell, \tag{4.6.20}$$

and

$$g_n^\ell = -\frac{1}{2}\sum_{(j,k)\in\{0,\ldots,\ell\}^2} Q(u_n^j, u_n^k) - \sum_{j\leq\ell} Q(u_n^j, \omega_n^\ell) - P(\omega_n^\ell \cdot \nabla\omega_n^\ell), \tag{4.6.21}$$

where

$$Q(m,n) = P(m\cdot\nabla n + n\cdot\nabla m), \tag{4.6.22}$$

P is orthogonal projector. We need the following Propositions to conclude the proof of the Theorem.

Proposition 4.6.4

With the notations (4.6.20) and (4.6.21), the following results are obtained : the sequence (f_n^ℓ) is bounded in E_∞ and

$$\lim_{\ell\to\infty}\lim_{n\to\infty} sup||g_n^\ell||_{L^2(\mathbb{R}^+, \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} = 0$$

and uniformly in ℓ

Proposition 4.6.5

Let (φ_n) be a divergence free sequence of vector fields in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, which is bounded and let φ^0 be a weak limit point of (φ_n). We have an integer j_0 and let $V^j = BF(\varphi^j)$ and $(\varphi^j)_{j\in\mathbb{N}}$ the profiles of the decomposition of φ_n. Then we obtain

$$\forall\ j \geq j_0, \quad V^j \in E_\infty \text{ as well as } \sum_{j\geq j_0} ||V^j||^2_{E_\infty} < +\infty$$

Proposition 4.6.6

Let $T \in \mathbb{R}^+ \cup \{+\infty\}$. There exists a constant C independent of T. Let (f_n) and (g_n) be two collections of bounded vector fields in E_T and in $L^2([0,T], \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$ respectively. If

$$sup_{n\in\mathbb{N}}||g_n||_{L^2([0,T],\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq Cexp(-2Csup_{n\in\mathbb{N}}||f_n||^4_{E_T})$$

$\exists!$ solution in E_T to the system:

$$\begin{cases} \partial_t r_n + P(r_n\cdot\nabla r_n) - \nu\Delta r_n + Q(r_n, f_n) &= g_n^\ell \quad \text{in } \mathbb{R}^+\times\mathbb{R}^3 \\ r_{n|t=0} &= 0 \end{cases} \tag{4.6.23}$$

we obtain

$$||r_n||_{E_T} \leq C||g_n||_{L^2([0,T],dotH^{-\frac{1}{2}}(\mathbb{R}^3))}(1 + exp(C||f||^4_{E_T}))$$

We postpone the proof of the Propositions and finish the proof of Theorem 4.6.1. By interpolation of the spaces $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$, $L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ and $L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ we know from Proposition 4.6.4 about uniform boundedness of (f_n^ℓ) in ℓ in the space $L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$. In applying Proposition 4.6.6 to the sequence r_n^ℓ, shows that for a large n , uniformly in ℓ, we obtain in relation to proposition 4.6.4

$$sup_{n\in\mathbb{N}}||g_n^\ell||_{L^2(\mathbb{R}^+,\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq Cexp\left(-2Csup_{(\ell,n)\in\mathbb{N}^2}||f_n^\ell||^4_{L^4(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))}\right),$$

hence we get

$$||r_n^\ell||_{E_\infty^\nu} \leq C||g_n^\ell||_{L^2(\mathbb{R}^+,\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))}\left(1 + exp\left(C||f_n^\ell||^4_{L^4(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))}\right)\right)$$

So (4.6.12) is proved. Next, the a priori estimate obtained is proved by contradiction: Since A is admissible, assume $\exists$ a sequence of global solutions of BFE in E_∞, with a bounded family of initial data φ_n) in a closed ball B^A_{BF} such that

$$\lim_{n\to\infty} ||BF(\varphi_n)||_{E_\infty^\nu} = +\infty \tag{4.6.24}$$

Theorem 3.3.2 applied to (φ_n). That implies $(BF(\varphi_n))$ is bounded in $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$, since it holds for each term of the decomposition. The contradiction is glaring, and the theorem is proved. $\square$

Proof of Proposition 4.6.6

The sequence (u_n^j) is bounded in E_∞ $\forall\ j \in \mathbb{N}$ since

$$||u_n^j||_{E_\infty} = ||V^j||_\infty.$$

For now, no a priori estimate for $||u_n^j||_{E_\infty}$ in more general B_{BF}^A case. From Proposition 4.6.5, we can deduce the bound on f_n^ℓ. And from Proposition 4.6.4, we have

$$\forall\ \ell \in \mathbb{N}, \quad ||\sum_{j\leq\ell} u_n^j||^2_{E_\infty} = \sum_{j\leq\ell} ||u_n^j||^2_{E_\infty} + o(1), \text{as } n \longrightarrow \infty$$

By the change of scale, we obtain

$$||\sum_{j\leq\ell} u_n^j||^2_{E_\infty} = \sum_{j\leq\ell} ||V_n^j||^2_{E_\infty} + o(1), \text{as } n \longrightarrow \infty$$

and hence for $V^j \in E_\infty$ $\forall\ j$ implies the boundedness of $\sum_{j\leq\ell} u_n^j$ in E_∞ uniformly in ℓ which proved the result about f_n^ℓ. Also g_n^ℓ is bounded since the sequences (u_n^j) and (w_n^j) uniform bound in j and ℓ due to the interpolation of the spaces

$$L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)),\ L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \text{ and } L^4(\mathbb{R}^3, L^4(\mathbb{R}^3))$$

. To give the proof of the limit on g_n^ℓ: It is sufficient to show the results as follows:

$$\lim_{\ell\to\infty} \lim_{n\to\infty} supQ(\sum_{j\leq\ell} u_n^j, w_n^\ell) = 0 \text{in} L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))$$

$$\lim_{\ell\to\infty} (\lim_{n\to\infty} supQ(w_n^\ell, w_n^\ell)) = 0 \text{in} L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3)).$$

According to Proposition 4.5.5,

$$\lim_{\ell\to\infty} \left(\lim_{n\to\infty} sup||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \right) = 0$$

$$\lim_{\ell\to\infty} (\lim_{n\to\infty} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} = 0 \tag{4.6.25}$$

It is sufficient to show that

$$\lim_{\ell\to\infty} \lim_{n\to\infty} sup||(\sum_{j\leq\ell} u_n^j + w_n^\ell)w_n^\ell||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0$$

But Holder's inequality yields

$$||(\sum_{j\leq\ell} u_n^j + w_n^\ell)w_n^\ell||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq ||\sum_{j\leq\ell} u_n^j + w_n^\ell||_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.6.26}$$

Since $\dot{H}^1(\mathbb{R}^3)$ is embedded into $L^6(\mathbb{R}^3)$, we have for all $n \in \mathbb{N}$,

$$|| \sum_{j\leq \ell} u_n^j + w_n^\ell ||_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3)} \leq C|| \sum_{j\leq \ell} u_n^j + w_n^\ell ||_{L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)}$$

$$\leq C|| \sum_{j\leq \ell} u_n^j ||_{E_\infty} + C||w_n^\ell||_{E_\infty}$$

by interpolation of spaces $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$, $L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ and $L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ and the Proposition one can infer that $(\sum_{j\leq\ell} u_n^j + w_n^\ell)$ is bounded in $L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))$, uniformly in ℓ and Proposition 4.6.6 is proved. $\square$

Proof of Proposition 4.6.5

For large j, the norm of φ^j in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is smaller than c, with

$$||\varphi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < c \Longrightarrow T_*(\varphi) = +\infty$$

Then we deduce that for large j ,

$$V^j \in E_\infty \text{ and } ||V^j||^2_{E_\infty} \leq 2||\varphi^j||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$$

But (3.3.4) connotes that the series of general term $||\varphi^j||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$ is convergent, therefore Proposition 4.5.5 follows. $\square$

Proof of Proposition 4.6.4

Any function in E_T is in $L^4([0,T], \dot{H}^1(\mathbb{R}^3))$, by interpolation of spaces $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$, $L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ and $L^4(\mathbb{R}^3, L^4(\mathbb{R}^3))$. Then the assumption

$$sup_{n\in\mathbb{N}}||g_n||_{L^2([0,T], \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq Cexp(-2Csup||f_n||^4_{E_T})$$

enables us to write that

$$||r_n||_{L^4([0,T], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} + ||r_n||_{L^2([0,T], \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} + ||r_n||_{L^4([0,T], L^4(\mathbb{R}^4))}$$
$$\leq C||g_n||_{L^2([0,T], \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} + C||r_n||_{L^4([0,T], \dot{H}^1(\mathbb{R}^3))} \left(||r_n||_{L^4([0,T], \dot{H}^1(\mathbb{R}^3))} + ||f_n||_{L^4([0,T], \dot{H}^1(\mathbb{R}^3))}\right)$$

So the result follows $\square$

4.7 Finite time singularities of solutions

In this section, solutions to the BFE in the critical homogenous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is considered.

In the theorems proved in previous sections, existence of weak solutions was obtained in

$$u \in L^\infty\left([0,T]; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T]; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \cap L^4\left([0,T], L^4(\mathbb{R}^3)\right)$$

That is when $f(u) = |u|^2 u$ which satisfies the estimate

$$sup_{0\leq t\leq T}||u||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\nu\int_0^T ||\nabla u(t)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}dt + 2\beta\int_0^T ||u(t)||^4|_{L^4}dt \leq ||u_0||^2_{\dot{H}^{\frac{1}{2}}} \tag{4.7.1}$$

We show that if the existence of initial data that leads to a finite time singularity is assumed, the set of such initial data is closed and sequentially compact modulo translations and dilations. This is a reminiscence of the work done by Walter Mieczyslaw Rusin in 2010 for Navier-Stokes equations. If one accepts smallness restrictions of initial data, solutions remain smooth globally in time. Let us thus define

$$\rho_{max} = sup\{\rho : T_{max}(\varphi) = +\infty \ for \ every \ \varphi \in \dot{H}^{\frac{1}{2}} \ with \ ||\varphi||_{\dot{H}^{\frac{1}{2}}} < \rho\} \tag{4.7.2}$$

corresponding $u(t,x)$ is a global solution of (BFE).

Global existence for any initial data would imply $\rho = +\infty$. It is assumed that the blow up in (BFE) is possible i.e $\rho < +\infty$. One could show that with the natural definition of the mild solution, the only reason ρ_{max} could be finite is the appearance of finite time singularities in the solution $u(t,x)$ for some initial data $\varphi(x)$

The aim of this section is to investigate the following question:

If ρ_{max} is finite does there exists an initial datum $\varphi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with

$$||\varphi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \rho_{max}$$

such that the solution $u(t,x)$ of the Cauchy problem BFE develops a singularity in finite time?

We show that the answer to this questions is affirmative. The initial data $\varphi(x)$ with $||\varphi||_{\dot{H}^{\frac{1}{2}}} = \rho_{max}$ leading to a singularity is called $\dot{H}^{\frac{1}{2}}$- minimal singularity-generating data. It is shown that if singularities exist, the set of the $\dot{H}^{\frac{1}{2}}$- minimal singularity-generating data is a nonempty subset of $\dot{H}^{\frac{1}{2}}$ which is compact, invariant the action of the scalings $\varphi(x) \to \lambda\varphi(\lambda x)$ and translations $\varphi(x) \to \varphi(x - x_0)$.

It is conceivable to have global solutions without an a-priori estimate

The only issue we want to address here is that for $u(t,x)$ not to be in E_∞ is by developing a singularity in finite time.

First, notice that it is well known that the embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ into L^3 is not compact. If we assume that the embedding is compact, one could consider a sequence of initial data $\varphi_n(x) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, leading to singular solutions, and such that

$||\varphi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \to C_B$ while $||\varphi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \geq C_B$.
One could extract a subsequence weakly converging to $\varphi^0(x)$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Then, compactness of the embedding would imply strong convergence of $\varphi_n(x)$ to $\varphi^0 \in L^3(\mathbb{R}^3)$
Under some very special circumstances, it is possible to reduce the situation to a sequence of initial data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for which the weak convergence actually turns out to be strong.
The following theorem unifies the theory of weak and mild solutions under the the assumption of the existence of initial data leading to finite time singularity. We show that the set of such initial data close and sequentially compact modulo translations and dilations.

Theorem 4.7.1

Suppose the following conditions hold:

(i) φ is an arbitrary function in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

(ii) $u(t,x)$ is a weak solution of (4.2.13)

(iii) $T_{max}(\varphi)$ is a maximal time of existence of a mild solution $v(t,x)$ in the sense of (1.1.7) with the same initial data φ

(iv) $M = \{\phi \in \dot{H}^{\frac{1}{2}} : T_{max}(\phi) < \infty, ||\phi||_{\dot{H}^{\frac{1}{2}}} = \rho_{max}\}$ is the set of initial data leading to singular solutions

(v) Assume that C_B in the sense of (3.5.8) is finite. That is $C_B \leq +\infty$

Then $M \neq \emptyset$

Proof

Let $\varphi_n(x) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ be sequence of initial data such that the corresponding mild solutions $v_n(t,x)$ become singular in finite time at (T_n, x_n) and $||\varphi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \searrow C_B$,

$$||\varphi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \geq C_G$$

If for some n we have

$$||\phi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = C_B$$

then we are done, thus we may assume that

$$||\varphi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} > C_B$$

Let $\lambda_n = \sqrt{T_n}$. For every n consider the re-scaled functions

$$(v_n(t,x))_{\lambda_n} = \lambda_n v_n(\lambda_n^2 t, \lambda_n x + x_n).$$

It is noted that $(v_n(t,x))_{\lambda_n}$ are again mild solution of BF and their $\dot{H}^{\frac{1}{2}}$-norm is preserved by the scaling. We can assume that all $u_n(t,x)$ become singular at $(1,0)$. We can further assume that $\varphi_n(x) \rightharpoonup \varphi^0(x)$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Since φ_n converges weakly in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, it is bounded and Theorem 3.3.2 yields that we can decompose the sequence as

$$\forall \ell \in \mathbb{N}\backslash\{0\}, \varphi_n(x) = \varphi^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j} \varphi^j \left(\frac{x - x_n^j}{h_n^j} \right) + \Psi_n^\ell(x) \tag{4.7.3}$$

with the properties (3.3.2) - (3.3.4). By weak convergence lower semi-continuity of the norm, we obtain

$$||\varphi^0||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)|} \leq C_B$$

Furthermore, it is noted that the functions φ^j are weak limit points in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ of the sequence

$$\varphi_n^j(x) = h_n^j \varphi_n(h_n^j x + x_n^j)$$

As a result of $\dot{H}^{\frac{1}{2}}$- norm scaling invariance, we obtain

$$||\varphi_n^j||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = ||\phi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$$

thus by the weak convergence lower semi-continuity of the norm, we obtain

$$||\varphi^j||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq C_B.$$

Note, that since $\phi_n(x)$ are divergence free and $\varphi^j(x)$ is a weak limit point of the sequence

$$\varphi_n^j(x) = h_n^j \varphi_n(h_n^j x + x_n^j)$$

this implies that $\varphi^j(x)$ are also divergence free .

If we pass to the limit $n \to \infty$, these two cases are considered.

Case 1 : There exists $j \in \{0, ..., \ell\}$ such that

$$||\varphi^j||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = C_B.$$

First, notice that since $||\varphi^j||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \searrow C_B$ then due to the fact that

$$\ell \in \mathbb{N}\backslash\{0\}, \; ||\varphi_n||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \sum_{j=0}^{\ell} ||\varphi^j||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + ||\Psi_n^\ell||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + o(1), \tag{4.7.4}$$

there can be at most one such j.

If $j = 0$ then assume that the corresponding solution $u(t,x)$ does not become singular. Notice that

$$||\varphi^0||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = C_B$$

implies that the sequence $\phi_n \rightharpoonup \varphi^0$ actually converges strongly in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $\phi_n \to \varphi^0$. This contradicts Theorem 3.5.6, since for n large we obtain

$$||\varphi^0 - \varphi_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \epsilon(u)$$

thus the solutions $u_n(t,x)$ belong in particular to E_2 but then by the Lazyzhenskaya-Prodi-Serrin condition, regular on $(0,2)$ which contradict the assumption that they become singular at $T = 1$. This implies that $\varphi^0(x)$ is the minimal data generating singularity. Thus we may assume $j \neq 0$. Property (3.3.4) implies that the decomposition of the initial data has the form

$$\varphi_n(x) = \frac{1}{h_n^j}\varphi^j\left(\frac{x - x_n^j}{h_n}\right) + \psi_n(x). \tag{4.7.5}$$

Since we have only one profile, we may drop the index j. We rescale equation (4.7.5) and obtain

$$h_n\varphi_n(h_nx + x_n) = \varphi(x) + h_n\psi_n(h_nx + x_n) \tag{4.7.6}$$

Notice that the rescaling preserves the $\dot{H}^{\frac{1}{2}}$)-norm. Without loss of generality, we may denote the rescale sequences $h_n\varphi_n(h_nx + x_n)$ by φ_n and $h_n\psi_n(h_nx + x_n)$ by $\psi_n(x)$, respectively. We are in a situation where the mild solutions $u_n(t,x)$, corresponding to initial data $\phi_n(x)$ become singular at (h_n^2, x_n).

Possibly passing to a subsequence we have the following cases:

(i) : $h_n \to +\infty$.

We have a sequence of solutions $u_n(t,x)$ regular on $(0, h_n^2)$ and developing a singularity at $T = h_n^2$. Since the $\dot{H}^{\frac{1}{2}}$-norm is preserved by the scaling, property (3.3.4) implies that $\psi_n(x)$ converges strongly to 0 in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Let

$$\epsilon > 0,\ ||w_0||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \frac{\epsilon}{2} \text{ and } u_0 \in L^2(\mathbb{R}^3)$$

This decomposition is independent of the index n since we scaled equation (4.7.5) to fix the profile φ Let $w_n(t,x)$ be the solution of (BF) with data $w_0(x) + \psi_n(x)$. For n large enough $w_n(t,x)$ is a global solution and satisfies the a-priori estimate

$$||w_n(t,\cdot)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\nu\int_0^t ||\nabla w_n(s,\cdot)||^2_{\dot{H}^{\frac{1}{2}}}ds + 2\beta\int_0^t ||w(t,\cdot)||_4^4 \leq ||w_0 + \psi_n||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \tag{4.7.7}$$

Since $\psi_n \to 0$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, we may choose n sufficiently large, so that

$$||w_n(t,\cdot)||^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \int_0^t ||\nabla w_n(s,\cdot)||^2_{\dot{H}^{\frac{1}{2}}} ds + \int_0^t ||w(t,\cdot)||_4^4 \leq \epsilon^2, \qquad (4.7.8)$$

which implies that for n large we have

$$sup_{t\geq 0}||w_n(t,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \epsilon \qquad (4.7.9)$$

Notice that $v_n(t,x) = u_n(t,x) - w_n(t,x)$ satisfy the equation

$$\begin{cases} \partial_t v_n - \nu\Delta v_n + v_n \cdot \nabla v_n + \beta|v_n|^2 v_n + v_n \cdot w_n + w_n \cdot \nabla v_n & \\ \qquad\qquad +\beta(|w_n|^2 + 2|v_n||w_n|)v_n & \\ \qquad\qquad +\beta(|v_n|^2 + 2|v_n||w_n|)w_n - \nabla q = 0 \text{ in } (0,h_n^2)\times\mathbb{R}^3, \\ \mathrm{div} v_n = 0 \text{in } (0,h_n^2)\times\mathbb{R}^3 \\ v_n(0,x) = v_0(x) \; in \; \mathbb{R}^3 \end{cases}$$

For every $T < h_n^2$, since

$$u_n(t,x) \in C\left([0,T], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T], \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \cap L^4\left([0,T], L^4(\mathbb{R}^3)\right)$$

and $w_n(t,x)$ are global, we have

$$v_n(t,x) \in C\left([0,T], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\right) \cap L^2\left([0,T], \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)\right) \cap L^4\left([0,T], L^4(\mathbb{R}^3)\right)$$

Furthermore, for every $t < h_n^2$ the energy estimate is obtained as follows

$$\begin{aligned} ||v_n(t,\cdot)||^2_{L^2}(\mathbb{R}^3) + 2\nu\int_0^t ||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)} ds + 2\beta\int_0^t ||v_n(s,\cdot)||^4_{L^4(\mathbb{R}^3)} \\ \leq ||v_0||^2_{L^2(\mathbb{R}^3)} + 2\left|\int_0^t\int_{\mathbb{R}^3} (v_n\cdot\nabla w_n)\cdot v_n dxds\right| \\ + 2\beta\left|\int_0^t\int_{\mathbb{R}^3} (|w_n|^2 + 2|v_n||w_n|)(v_n^2)_x dxds\right| \\ + 2\left|\int_0^t\int_{\mathbb{R}^3} \beta(|v_n|^2 + 2|v_n||w_n|)w_n\partial_x v_n dxds\right| \end{aligned}$$

To estimate the 2nd term on the rhs of the above energy estimate due to Holder's inequality and Sobolev imbeddings we have the following

$$\begin{aligned} \left|\int_0^t\int_{\mathbb{R}^3} (v_n\cdot\nabla w_n)\cdot v_n dxds\right| &= \left|\int_0^t\int_{\mathbb{R}^3} (v_n\otimes w_n)\cdot\nabla v_n dxds\right| \\ &\leq C\int_0^t ||w_n(s,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)} ds \end{aligned}$$

From (4.7.9) we have

$$\int_0^t ||w_n(s,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)} ds \leq C\epsilon\int_0^t ||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)} ds \qquad (4.7.10)$$

To estimate the other terms, we have

$$\left|\int_0^t \int_{\mathbb{R}^3} \beta(|w_n|^2 + 2|v_n||w_n|)(v_n^2)_x dxds\right|$$

$$\leq C\int_0^t ||w_n||^2_{\dot{H}^{\frac{1}{2}}}||v_n||^2_{\dot{H}^1} + ||v_n||_{L^2}||w_n||_{\dot{H}^{\frac{1}{2}}}||v_{n,x}||^2_{\dot{H}^1}ds$$

$$\leq C\epsilon\int_0^t \epsilon||v_n||^2_{\dot{H}^1} + ||v_n||_{L^2}||v_n||^2_{\dot{H}^1}ds$$

$$\left|\int_0^t \int_{\mathbb{R}^3} \beta(|v_n|^2 + 2|v_n||w_n|)w_n(v_{n,x})dxds\right|$$

$$\leq C\int_0^t ||v_n||^3_{L^2(\mathbb{R}^3)}||w_n||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$$

$$+ 2||v_n||_{L^2(\mathbb{R}^3)}||w_n||^2||v_{n,x}||_{\dot{H}^1(\mathbb{R}^3)}ds$$

$$\leq C\epsilon\int_0^t ||v_n||^3_{L^2(\mathbb{R}^3)} + 2\epsilon||v_n||_{L^2(\mathbb{R}^3)}||v_{n,}||_{\dot{H}^1}ds$$

Thus we have

$$||v_n(t,\cdot)||^2_{L^2(\mathbb{R}^3)}+2\nu\int_0^t ||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)}ds+2\beta\int_0^t ||v_n(s,\cdot)||^4_{L^4}ds \leq ||v_0||^2_{L^2(\mathbb{R}^3)}$$

$$+ 2C\epsilon\int_0^t ||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)}ds + 2C\epsilon\int_0^t \epsilon||v_n||^2_{\dot{H}^1} + ||v_n||_{L^2}||v_n||^2_{\dot{H}^1}ds$$

$$+ 2C\epsilon\int_0^t ||v_n||^3_{L^2(\mathbb{R}^3)} + 2\epsilon||v_n||_{L^2(\mathbb{R}^3)}||v_{n,x}||_{\dot{H}^1}ds$$

Take $\epsilon < min(\frac{1}{2C}, \frac{C_G}{2})$ and as $\epsilon \to 0$ We obtain for all $t < h_n^2$

$$||v_n(t,\cdot)||^2_{L^2(\mathbb{R}^3)} + 2\nu\int_0^t ||v_n(s,\cdot)||^2_{\dot{H}^1(\mathbb{R}^3)}ds + 2\beta\int_0^t ||v_n(s,\cdot)||^4_{L^4(\mathbb{R}^3)}$$

$$\leq ||v_0||^2_{L^2(\mathbb{R}^3)}ds \quad (4.7.11)$$

Interpolating of the spaces

$$L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)), L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \text{ and } L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$$

we obtain that $v_n(t,x) \in L^4([0,t], \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ for all $T < h_n^2$. Assume that for every n and every $t \in [0, \frac{h_n^2}{2}]$ we have $||v_n(t,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \geq \epsilon$. This implies

$$\epsilon^4\frac{h_n^2}{2} \leq \int_0^{\frac{h_n^2}{2}} ||v_n(s,\cdot)||^4_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}ds \leq ||v_0||^4_{L^2(\mathbb{R}^3)} \quad (4.7.12)$$

Finally we notice that for n sufficiently large, inequality (4.7.12) cannot hold, thus for n large enough there exist $0 < t_n < h_n^2$ such that $||v_n(t_n,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \epsilon$. In this case however, because of

$$u_n(t,x) = v_n(t,x) + w_n(t,x)$$

, we have

$$||u_n(t_n,\cdot)||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < C_B \quad (4.7.13)$$

thus $u_n(t,x)$ is a global solution. Hence we have obtained a contradiction.

(ii) : $h_n \to 0$

Since ψ_n converges to zero strongly in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ we have strong convergence $\varphi_n \to \varphi$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and solutions corresponding to data φ_n become singular at times $h_n^2 \to 0$. Thus assuming that $\varphi(x)$ leads to a global solution $u(t,x)$, for n large enough we have $||\varphi_n - \varphi||_{\dot{H}^{\frac{1}{2}}} < \epsilon(u)$. However inequality (3.5.7) does not hold. This implies that $\varphi(x)$ is the singularity-generating data

(iii) : $h_n \to h$ with $h \neq 0$ and $h < +\infty$

As in the case above, we have strong convergence of initial data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, however the stability estimate is violated. This implies that $\varphi(x)$ is the minimal singularity-generating data.

Case 2 : For all $j \in \{0, ..., \ell\}$ we have $||\varphi^j||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < C_G$ and

$$\varphi_n(x) = \varphi^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j}\varphi^j\left(\frac{x - x_n^j}{h_n^j}\right) + \psi_n^\ell(x)$$

Let V^j be the mild solution of (BFE) corresponding to data φ^j for $j \in \{0, ..., \ell\}$. It is noted that from the the assumption

$$||\varphi^j||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < C_G$$

the solutions are global and $V^j \in E_\infty$. From Theorem 4.6.1, setting

$$u_n(x) = V^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j}V^j\left(\frac{t}{(h_n)^2}, \frac{x - x_n^j}{h_n^j}\right) + w_n^\ell(t,x) + r_n^\ell(t,x) \tag{4.7.14}$$

where $w_n^\ell(t,x)$ is a solution of homogeneous heat equation with initial data $\psi_n^\ell(x)$ and $r_n^\ell(t,x)$ satisfy

$$\begin{cases} \partial_t r_n^\ell + P(r_n^\ell \cdot \nabla r_n^\ell) - \nu\Delta r_n^\ell + Q(r_n^\ell, f_n^\ell) &= g_n^\ell \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ r_{n|t=0}^\ell &= 0 \end{cases} \tag{4.7.15}$$

where

$$f_n^\ell = \sum_{j \leq \ell} u_n^j + \omega_n^\ell, \tag{4.7.16}$$

and

$$g_n^\ell = -\frac{1}{2}\sum_{(j,k)\in\{0,\dots,\ell\}^2} Q(u_n^j, u_n^k) - \sum_{j\leq\ell} Q(u_n^j, \omega_n^\ell) - P(\omega_n^\ell \cdot \nabla\omega_n^\ell), \tag{4.7.17}$$

and u_n^ℓ are defined as

$$u_n^j(t,x) = \frac{1}{h_n^j} V^j \left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j} \right) \tag{4.7.18}$$

the terms on the rhs of (4.7.14) but likely $r_n^\ell(t,x)$ are in E_∞. The issue now is that $u_n^\ell(t,x)$ do not satisfy the energy estimate as a necessity and thus may possibly create a problem for the application of Proposition 4.6.6. The norm of φ^j must be very small and it is possible that $C_E < C_G$. Absence of the energy estimate may lead to lack of control over $||f_n^\ell||_{E_\infty}$ with increasing ℓ (which ensures possibility of Proposition 4.6.6). However, due to 3.3.4 there exists j_0 which depends on C_E and C_G only such that

$$||\varphi^j|| < C_E \quad \text{for } j \geq j_0$$

Thus only a finite number of V^j is obtained which fail to satisfy the energy estimate. This, together with property 3.3.4 and the norms scaling invariance, permits us to infer that

$$\sum_{j=0}^{\infty} ||V^j||_{E_\infty}^2 < +\infty \tag{4.7.19}$$

and it is going to play a crucial role later and takes care of the issue.

If it can be shown that for ℓ, n large enough, $r_n^\ell(t,x) \in E_2$, it shows that $u_n(t,x) \in E_2$ hence $u_n \in L^5((0,2) \times \mathbb{R}^3)$ and by Ladyzhenskaya-Prodi-Serrin condition are regular up to $T = 2$. This is a contradiction to the assumption that solutions corresponding to initial data φ_n become singular at $T = 1$. The terms in rhs of (4.7.15) are all global except $r_n^\ell(t,x)$, attention will be drawn to this term. In order to control $r_n^\ell(t,x)$, Proposition 4.6.6 will be used. It is sufficient to show that $||f_n^\ell||_{E_2}$ is bounded uniformly in ℓ and n and that for ℓ and n large enough, we have $||g_n^\ell||_{L^2([0,2],\dot{H}^{-\frac{1}{2}})}$ small enough that condition on g_n in Proposition 4.6.6 is satisfied. Avoidance of issues with length scaling of the time intervals on which $\mathbb{R}^+$ is considered. The sequence ψ_n^ℓ yielded is bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ uniformly in ℓ and w_n^ℓ bounded in E_∞ uniformly in ℓ by $||\psi_n^\ell||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$. Hence there is also a bound in E_2. As a result the a priori estimate in $\dot{H}^{\frac{1}{2}}$ for a very small data and the scale invariance of the norms, we obtain the estimate

$$||u_n^\ell||_{E_\infty} \leq C||\varphi^j||\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \;\text{ for } j \geq j_0 \tag{4.7.20}$$

Proposition 4.6.6 gives

$$||\sum_{j \leq \ell} u_n^j||_{E_\infty}^2 = \sum_{j \leq \ell} ||u_n^j||^2 o(1), \quad n \to \infty \tag{4.7.21}$$

thus property (4.7.10) with

$$||u_n^j||_{E_\infty} = ||V^j||_{E_\infty}$$

gives

$$||\sum_{j\leq \ell} u_n^j||_{E_2}^2 \leq ||\sum_{j\leq \ell} u_n^j||_{E_\infty}^2 \leq \sum_{j=0}^{j_0} ||V^j||_{E_\infty} + 2||\varphi_n||^2_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)}$$

$$\leq \sum_{j=0}^{j_0} ||V^j||_{E_\infty} + 2C_G \quad (4.7.22)$$

for a very large n. j_0 is independent of ℓ and n. In a particular case, there is an estimate on $||f_n^\ell||_{E_\infty}$ uniform in ℓ which implies a uniform estimate of $||f_n^\ell||_{E_2}$. It can be shown that g_n^ℓ is small in $L^2([0,2], \dot H^{\frac{1}{2}}(\mathbb{R}^3))$ for a large n. The terms u_n^ℓ, w_n^ℓ are in E_∞ due to assumption on initial data and it is noted that

$$||\varphi^j||_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)} < C_G \text{ for } j \in \{j_0, ..., \ell\}$$

By interpolation of spaces $L^\infty(\mathbb{R}^+, \dot H^{\frac{1}{2}}(\mathbb{R}^3))$, $L^2(\mathbb{R}^+, \dot H^{\frac{3}{2}}(\mathbb{R}^3))$ and $L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$, it is obtained that all u_n^ℓ and w_n^ℓ belong to $L^{\frac{8}{3}}(\mathbb{R}^+, \dot H^{\frac{5}{4}}(\mathbb{R}^3))$ and for $j \geq j_0$, estimates are obtained by norms of initial data $||\varphi^j||_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)}$. For w_n^ℓ, an a priori estimate is obtained by $||\psi_n^\ell||_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)}$. The product rule of Sobolev spaces is recalled as follows

$$\forall\ s,t < \frac{3}{2},\ s+t > 0,\ \ ||ab||_{\dot H^{s+t-\frac{3}{2}}(\mathbb{R}^3)} \leq C(s,t)||a||_{\dot H^s(\mathbb{R}^3)}||b||_{\dot H^t(\mathbb{R}^3)} \qquad (4.7.23)$$

The building blocks of g_n^ℓ are of the form of $P(a_n \cdot \nabla b_n)$ where

$$a_n,\ b_n \in L^{\frac{8}{3}}(\mathbb{R}^+, \dot H^{\frac{5}{4}}(\mathbb{R}^3))$$

thus we have

$$||P(a_n \cdot \nabla b_n)||_{L^{\frac{3}{4}}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq C||a_n||_{L^{\frac{8}{3}}(\mathbb{R}^+, \dot H^{\frac{5}{4}}(\mathbb{R}^3))}||b_n||_{L^{\frac{8}{3}}(\mathbb{R}^+, \dot H^{\frac{5}{4}}(\mathbb{R}^3))} \qquad (4.7.24)$$

Hence we obtain

$$||Q(u_n^j, u_n^k)||_{L^{\frac{3}{4}}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq ||u_n^j||_{E_\infty}||u_n^k||_{E_\infty}, \qquad (4.7.25)$$

$$||Q(u_n^j, w_n^\ell)||_{L^{\frac{3}{4}}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq C||u_n^j||_{E_\infty}||\psi_n^\ell||_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)}, \qquad (4.7.26)$$

$$||P(w_n^\ell \cdot \nabla w_n^\ell)||_{L^{\frac{3}{4}}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq C||\psi_n^\ell||_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)} \qquad (4.7.27)$$

Estimates (4.7.24)- (4.7.27) show that g_n^ℓ is bounded in $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$ for the bound that is uniform in ℓ because of the inequality $2ab \leq a^2 + b^2$ the a priori estimate for nearly all u_n^ℓ and property(3.3.4). Those estimates are also uniform in n because of scaling invariance of norm involved. If g_n^ℓ can be made small for large n in $L^4(\mathbb{R}^+, \dot H^{-1}(\mathbb{R}^3))$ uniformly in ℓ then by interpolation between spaces

$L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))$ and $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$, small g_n^ℓ is obtained in $L^2(\mathbb{R}^+, \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$ and also in $L^2([0,2], \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$ For very large n the following quantities

$$||Q(u_n^j, u_n^k)||_{L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))} \tag{4.7.28}$$

$$||Q(\sum_{j\leq \ell} u_n^j, w_n^\ell)||_{L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))} \tag{4.7.29}$$

$$||P(w_n^\ell \cdot \nabla w_n^\ell)||_{L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))} \tag{4.7.30}$$

become very small for ℓ fixed. Because of the divergence free condition, this reduces to showing that

$$\forall j \neq k, \ \lim_{n\to\infty} ||u_n^j u_n^k||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0 \tag{4.7.31}$$

$$\lim_{\ell\to\infty} \lim sup_{n\to\infty} ||w_n^\ell \sum_{j\leq \ell} u_n^j||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0 \tag{4.7.32}$$

and

$$\lim_{\ell\to\infty} \lim sup_{n\to\infty} ||w_n^\ell w_n^\ell||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0 \tag{4.7.33}$$

For (4.7.33), we have

$$||w_n^\ell w_n^\ell||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq ||w_n^\ell||_{L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.7.34}$$

By the imbedding

$$\dot{H}^1 \subset L^6(\mathbb{R}^3)$$

we obtain

$$||w_n^\ell||_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \leq C ||w_n^\ell||_{L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.7.35}$$

Interpolation of the spaces $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$, $L^2(\mathbb{R}^+, \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ and $L^4(\mathbb{R}^+, L^4(\mathbb{R}^3))$ gives the bound of w_n^ℓ in $L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^+))$ and combined with the a priori estimate for heat equation and an estimate uniform in n is obtained. An estimate of w_n^ℓ is also obtained in $L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$ by $||\psi_n^\ell||$ which can be made as small as possible for large n and ℓ. This implies an appropriate estimate in $L^4([0,2], \dot{H}^{-1}(\mathbb{R}^3))$ of $P(w_n^\ell \cdot \nabla w_n^\ell)$. The same pattern is followed by estimate (4.7.32). We obtain

$$||w_n^\ell \sum_{j\leq \ell} u_n^j||_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq || \sum_{j\leq \ell} u_n^j||_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.7.36}$$

and by the embedding

$$\dot{H}^1 \subset L^6(\mathbb{R}^3)$$

we obtain

$$|| \sum_{j\leq \ell} u_n^j||_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \leq C || \sum_{j\leq \ell} u_n^j||_{L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))} ||w_n^\ell||_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.7.37}$$

By interpolation, we obtain

$$||\sum_{j\leq\ell} u_n^j||_{L^4(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))}$$
$$\leq ||\sum_{j\leq\ell} u_n^j||^{\frac{1}{3}}_{L^4(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} ||\sum_{j\leq\ell} u_n^j||^{\frac{1}{3}}_{L^4(\mathbb{R}^+,\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} ||\sum_{j\leq\ell} u_n^j||^{\frac{1}{3}}_{L^4(\mathbb{R}^+,L^4(\mathbb{R}^3))} \quad (4.7.38)$$

On the rhs, advantage of $C_{NS} = C_G$ is taken and we obtain

$$||\sum_{j\leq\ell} u_n^j||^{\frac{1}{3}}_{L^4(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} ||\sum_{j\leq\ell} u_n^j||^{\frac{1}{3}}_{L^4(\mathbb{R}^+,\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} ||\sum_{j\leq\ell} u_n^j||^{\frac{1}{3}}_{L^4(\mathbb{R}^+,L^4(\mathbb{R}^3))}$$
$$\leq ||\sum_{j\leq\ell} u_n^\ell||_{E_\infty} \quad (4.7.39)$$

Estimate (4.7.31) gives a uniform bound in ℓ and n. From (3.3.3) applied to the second term on the rhs of (4.7.38) we obtain (4.7.33) and hence an estimate of $Q(\sum_{j\leq\ell} u_n^j, w_n^\ell)$ in

$$L^2([0,2], \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$$

The estimate is summed over

$$j,\ k \in \{0, ..., \ell\},\ j \neq k$$

thus have no control as ℓ increases. However, once ℓ is fixed to make $||\psi_n^\ell||$, a large n can be considered and the sum over $j \neq k$ is still arbitrarily small. Hence we obtain, for fixed ℓ

$$\lim_{n\to\infty} ||\sum_{j\neq k, j,k\in\{0,\ldots,\ell\}} Q(u_n^j, u_n^k)||_{L^2([0,2],\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} = 0 \quad (4.7.40)$$

Thus we obtain

$$\lim_{\ell\to\infty} \lim sup_{n\to\infty} ||g_n^\ell||_{L^2[0,2],\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} = 0 \quad (4.7.41)$$

Combination of this with above shows uniform estimate of $||f_n^\ell||_{E_2}$ and Proposition 4.6.6 gives us a uniform bound on $||r_n^\ell||_{E_2}$ thus also a bound of $||u_n||_{E_2}$ uniform in n. Applying the Ladyzhenskaya-Prodi-Serrin condition, we have that the solution $u_n(t,x)$ is regular up to time $T = 2$ which contradicts the assumption that the solution u_n becomes singular at the time $T = 1$. This rules out case 2. This concludes the proof. □

Chapter 5
SUMMARY AND CONCLUSIONS

5.1 Introduction

This chapter presents the summary of the study, conclusions and direction of the future work. Some limitations are also identified. The research aimed to establish the existence of solutions of BFE in homogeneous Sobolev spaces for different conditions imposed on the damping term. Also, stability results of the system were obtained in the critical homogeneous Sobolev spaces and some other qualitative properties. The BFE is an .....

5.2 Summary

The focus of this research was to achieve the following objectives:

(i) To obtain existence results for weak solutions of BFE in a homogeneous Sobolev space when the damping term $f(u)$ is continuous, continuously differentiable and satisfies Lipschitz condition.

(ii) To analyze how Profile Decomposition is propagated by BFE.

(iii) To obtain stability results of the system in a critical space and other qualitative properties of solution.

(iv) To investigate the possibility of finite time singularities of solutions for BFE in a critical Sobolev space.

The above objectives were achieved by the answers given to the following research questions:

(1) Given initial data φ in homogeneous Sobolev space $\dot{H}^s$, can the existence results of solution of BFE associated with the initial conditions be obtained when the damping term $f(u)$ is continuous, continuously differentiable and satisfies Lipschitz condition?

(2) If B_{BF}^{A} is a ball in L^3 with center zero and the elements of $\dot{H}^{\frac{1}{2}} \cap B_{BF}^{L^3}$ generate global solutions of BFE, can a priori estimates be obtained for those solutions given the sequences of solutions in $\mathbb{R}^3$ associated with bounded sequences of initial data in $\dot{H}^{\frac{1}{2}}$?

(3) If $\rho_{max} < \infty$ can there be an initial datum $\varphi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $||\varphi||_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \rho_{max}$, such that the solution $u(t,x)$ of BFE develops a singularity in finite time?

These questions were answered in affirmative.

5.2.1 Limitations

Some limitations are identified:

- The areas of applications of BFE is limited to non-Newtonian fluids.
- Some qualitative properties of solution of BFE achieved was due to scale-invariant property of the equation which is only possible at the critical value of the exponent in the equation.

5.3 Future work

Based on the analysis of the qualitative properties of solutions of BFE, there are several ways one can extend this research work presented in this thesis for global solutions of the equation. Let us recall the chain of spaces

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)\ |_{p<+\infty} \hookrightarrow \nabla BMO(\mathbb{R}^3)$$

In that chain of spaces, $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ stands for a homogeneous Besov space.

1. Asymptotics as well as stability of solutions to the BFE for critical Besov space

2. Blow-up of critical Besov norm of solution to the BFE for potential blow up time, T satisfying some necessary conditions.

3. Regularity of solution of BFE in critical spaces.

4. If a sequence of solutions of BFE is given in $\mathbb{R}^3$, with a sequence of initial data bounded in $\dot{H}^{\frac{1}{2}}$, it can be shown that the decomposition of solution to a sum of orthogonal profiles is possible and it will be bounded in $\dot{H}^{\frac{1}{2}}$, up to a remainder term small in homogeneous Besov space $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. Based on the embedding theorem between the spaces. Also if B is the largest open ball in $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ such that $\dot{H}^{\frac{1}{2}} \cap B$ generate global solutions, then an a priori estimate can be obtained for those solutions.

5. If a sequence of solutions of BFE is given in $\mathbb{R}^3$, with a sequence of initial data bounded in $\dot{H}^{\frac{1}{2}}$, it can be shown that the decomposition of solution to a sum of orthogonal profiles is possible and it will be bounded in $\dot{H}^{\frac{1}{2}}$, a remainder term small in $\nabla BMO(\mathbb{R}^3)$. Based on the embedding theorem between the spaces. Also if B is the largest open ball in $\nabla BMO(\mathbb{R}^3)$ such that $\dot{H}^{\frac{1}{2}} \cap B$ generate global solutions, then an a priori estimate can be obtained for those solutions.

6. Another important problem is to unify the theory of mild solutions and Leray-Hopf weak solutions in the homogeneous Besov space $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ and $\nabla BMO(\mathbb{R}^3)$

7. Another interesting area to explore is the addition of 'noise' (stochastic process) to the equation with some underling probability space.

Elements of Besov space satisfy $\dot{B}^s_{p,\infty}(\mathbb{R}^3)$ satisfy

$$||u||_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} = sup_{j\in\mathbb{Z}} 2^{js} ||\Delta_j u||_{L^p(\mathbb{R}^3)} < +\infty,$$

where $\varphi \in C_c^\infty([\frac{1}{2}, 2])$

$\nabla BMO(\mathbb{R}^3)$ denotes the space of first derivative functions in $BMO(\mathbb{R}^3)$. The norm

$||u||_{BMO(\mathbb{R}^3)}$can be defined as follows

$$||u||_{BMO(\mathbb{R}^3)} = sup_{x_0,R}\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}|u - u_{B(x_0,R)}|dx,$$

where

$$u_{B(x_0,R)} = \frac{1}{|B(x_0,R)|}\int_{B(x_0,R)}u(x)dx$$

.

Another area of future research that the study BFE can take is developing a meshless method based on radial basis functions in a finite difference mode (RBF-FD). RBF is a meshless method for solving fluid flow problems have become a promising alternative to mesh-based methods like finite difference method, finite element method etc. The RBFs that are likely to develop for BFE and expression of $\phi(r)$ are as follow:

Multi-quadratic (MQ):	$\phi(r) = \sqrt{r^2 + \sigma^2}$
Inverse Multi-quadratic (IMQ):	$\phi(r) = \frac{1}{\sqrt{r^2 + \sigma^2}}$
Inverse Quadratic (IQ):	$\phi(r) = \frac{1}{(r^2 + \sigma^2)}$
Gaussian (GA):	$\phi(r) = exp(-(r\sigma)^2)$

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